Finite Fields and Their Applications 28 (2014) 140–147

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

On restricted sumsets over a field

Lilu Zhao

School of Mathematics, Hefei University of Technology, Heifei 230009, People's Republic of China

ARTICLE INFO

Article history:
Received 19 August 2013
Received in revised form 8 December 2013
Accepted 7 February 2014
Available online 6 March 2014
Communicated by Igor Shparlinski

MSC:
primary 11B13
secondary 11C08, 11T06

Keywords:
Sumset
Field
The polynomial method

ABSTRACT

We consider restricted sumsets over field F. Let

$$C = \{a_1 + \cdots + a_n\colon\ a_1 \in A_1, \ldots, a_n \in A_n,\ a_i - a_j \notin S_{ij} \\ \text{if } i \neq j\},$$

where S_{ij} $(1 \leqslant i \neq j \leqslant n)$ are finite subsets of F with cardinality m, and $A_1, \ldots, A_n$ are finite nonempty subsets of F with $|A_1| = \cdots = |A_n| = k$. Let $p(F)$ be the additive order of the identity of F. It is proved that $|C| \geqslant \min\{p(F), n(k-1) - mn(n-1) + 1\}$ if $p(F) > mn$. This conclusion refines the result of Hou and Sun [11].

1. Introduction

Let F be a field. Denote by $p(F)$ the additive order of the identity of F. It is well-known that $p(F)$ is either infinite or a prime. For a finite set A, we use A to denote the cardinality of A.

Suppose that $A_1, \ldots, A_n$ are finite nonempty subsets of F with $|A_j| = k_j$ for $1 \leqslant j \leqslant n$. The Cauchy–Davenport theorem asserts that

$$\bigl|\{a_1 + \cdots + a_n\colon\ a_1 \in A_1, \ldots, a_n \in A_n\}\bigr| \geqslant \min\bigl\{p(F),\ k_1 + \cdots + k_n - n + 1\bigr\}.$$

E-mail address: zhaolilu@gmail.com.

http://dx.doi.org/10.1016/j.ffa.2014.02.004
1071-5797/

Let A be a finite subset of F. We define

$$n^\wedge A = \{a_1 + \cdots + a_n\colon\ a_1, \ldots, a_n \in A,\ a_1, \ldots, a_n \text{ are distinct}\}.$$

P. Erdös and H. Heilbronn [7] conjectured that

$$\left|2^\wedge A\right| \geqslant \min\{p(F),\ 2|A| - 3\}.$$

This conjecture was solved by Dias da Silva and Hamidoune [5], who established

$$\left|n^\wedge A\right| \geqslant \min\{p(F),\ n|A| - n^2 + 1\}.$$

In 1995–1996, Alon, Nathanson and Ruzsa [2,3] developed the polynomial method to show that if $0 < k_1 < k_2 < \cdots < k_n$, then

$$\left|\{a_1 + \cdots + a_n\colon\ a_i \in A_i,\ a_1, \ldots, a_n \text{ are distinct}\}\right| \geqslant \min\left\{p(F),\ \sum_{j=1}^{n}(k_j - j) + 1\right\}.$$

Various restricted sumsets of $A_1, \ldots, A_n$ were investigated in [11–16]. In particular, Hou and Sun [11] considered the following sumset

$$C = \{a_1 + \cdots + a_n\colon\ a_1 \in A_1, \ldots, a_n \in A_n,\ a_i - a_j \notin S_{ij} \text{ if } i \neq j\}, \tag{1.1}$$

where S_{ij} $(1 \leqslant i \neq j \leqslant n)$ are finite subsets of F. Hou and Sun [11] established the following result.

Theorem 1.1 *(Hou–Sun). Let C be given by (1.1) with $|S_{ij}| = m$ $(1 \leqslant i \neq j \leqslant n)$. If $|A_1| = \cdots = |A_n| = k$ and $p(F) > \min\{n(k-1) - mn(n-1),\ mn\}$, then*

$$|C| \geqslant n(k-1) - mn(n-1) + 1.$$

The first result of this paper is to refine Theorem 1.1.

Theorem 1.2. *Let S_{ij} $(1 \leqslant i \neq j \leqslant n)$ be finite subsets of F with cardinality m, and let C be defined in (1.1). Suppose that $|A_j| \in \{k, k+1\}$ for $1 \leqslant j \leqslant n$. If $p(F) > \min\{mn,\ \sum_{j=1}^{n}(|A_j| - 1) - mn(n-1)\}$, then*

$$|C| \geqslant \sum_{j=1}^{n}\left(|A_j| - 1\right) - mn(n-1) + 1. \tag{1.2}$$

It was pointed out by Hou and Sun [11] that when $p(F) \leqslant n(k-1) - mn(n-1)$, Theorem 1.1 only implies

$$|C| \geqslant n\left\lfloor \frac{p(F) - 1}{n} \right\rfloor + 1, \tag{1.3}$$