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On L-functions of certain exponential sums

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ABSTRACT

Let $\mathbb{F}_q$ denote the finite field of order q of characteristic p. We study the p-adic valuations for zeros of L-functions associated with exponential sums of the following family of Laurent polynomials

$$f(x) = a_1x_{n+1}\left(x_1 + \frac{1}{x_1}\right) + \cdots + a_nx_{n+1}\left(x_n + \frac{1}{x_n}\right) + a_{n+1}x_{n+1} + \frac{1}{x_{n+1}}$$

where $a_i \in \mathbb{F}_q^*$, $i = 1, 2, \ldots, n + 1$. When $n = 2$, the estimate of the associated exponential sum appears in Iwaniec's work on small eigenvalues of the Laplace–Beltrami operator acting on automorphic functions with respect to the group $\Gamma_0(p)$, and Adolphson and Sperber gave complex absolute values for zeros of the corresponding L-function. Using the decomposition theory of Wan, we determine the generic Newton polygon (q-adic values of the reciprocal zeros) of the L-function. Working on the chain level version of Dwork's trace formula and using Wan's decomposition theory, we are able to give an explicit Hasse polynomial for the generic Newton polygon in low dimensions, i.e., $n \leqslant 3$.

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1. Introduction

L-functions have been a powerful tool to investigate exponential sums in number theory. To estimate an exponential sum, people are interested in the zeros and poles of the corresponding L-function. Mathematicians study the number of these zeros and poles, the complex absolute values, l-adic absolute values of them for primes $l \neq p$, and p-adic absolute values of them, especially for some interesting varieties and exponential sums [6,7,13–15]. Deligne's theorem gives the general information for complex absolute values of the zeros and poles of L-function. For l-adic absolute values, it is well-known that if $l \neq p$ then all the zeros and poles have l-adic absolute value 1. However, for p-adic absolute values, it is still very mysterious [9–12,19,20], especially in higher dimensions.

In this paper, we consider the following family of Laurent polynomials

$$f(x_1, \dots, x_{n+1}) = a_1 x_{n+1}\left(x_1 + \frac{1}{x_1}\right) + \dots + a_n x_{n+1}\left(x_n + \frac{1}{x_n}\right) + a_{n+1}x_{n+1} + \frac{1}{x_{n+1}} \tag{1.1}$$

where $a_i \in \mathbb{F}_q^*$, $i = 1, 2, \dots, n+1$. The exponential sum associated to f is defined to be

$$S_k^*(f) = \sum_{x_1,\dots,x_{n+1} \in \mathbb{F}_{q^k}^*} \zeta_p^{\mathrm{Tr}_k f(x_1,\dots,x_{n+1})},$$

where ζ_p is a fixed primitive p-th root of unity in the complex numbers and Tr_k denotes the trace map from the k-th extended field $\mathbb{F}_{q^k}$ to the prime field $\mathbb{F}_p$. When $n = 2$, the estimate of exponential sum $S_k^*(f)$ is vital in analytic number theory. It appears in Iwaniec's work [8] on small eigenvalues of the Laplace–Beltrami operator acting on automorphic functions with respect to the group $\Gamma_0(p)$. The main result in [8] on the improvement about the low bound of small eigenvalues relies on the estimate for a class of bilinear forms in certain Kloosterman sums. And the estimate was reduced to estimating the exponential sums $S_k^*(f)$ for Laurent polynomials f of the form (1.1) in three variables. Adolphson and Sperber studied the complex absolute value estimate for the exponential sums. In this paper, the p-adic arithmetic property of the exponential sums is studied.

To understand the sequence $S_k^*(f) \in \mathbb{Q}(\zeta_p)$ $(1 \leqslant k < \infty)$ of algebraic integers, we study the L-function associated to $S_k^*(f)$

$$L^*(f, T) = \exp\left(\sum_{k=1}^{\infty} S_k^*(f)\frac{T^k}{k}\right).$$

By the theorem of Adolphson and Sperber [1], the L-function $L^*(f,T)^{(-1)^n}$ for non-degenerate f in $n+1$ variables is a polynomial of degree $(n+1)!\mathrm{Vol}(\Delta)$, where $\Delta = \Delta(f)$ is the Newton polyhedron of f defined explicitly later. As the origin is an interior point of the Newton polyhedron Δ, by the theorem of Denef and Loeser [5], we have

Theorem 1.1. *Assume that f is non-degenerate, that is,*

$$\prod_{(c_1,c_2,\dots,c_n)\in\{\pm 1\}^n} (2c_1a_1 + 2c_2a_2 + \dots + 2c_na_n + a_{n+1}) \neq 0.$$

Then, the L-function $L^*(f,T)$ associated to the exponential sum $S_k^*(f)$ is pure of weight $n+1$, i.e.,

$$L^*(f,T)^{(-1)^n} = \prod_{i=1}^{(n+1)!\mathrm{Vol}(\Delta(f))} (1 - \alpha_i T)$$

with the complex absolute value $|\alpha_i| = q^{(n+1)/2}$.