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Hyperoval constructions on the Hermitian surface

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ABSTRACT

New infinite families of hyperovals of the generalized quadrangle $\mathcal{H}(3,q^2)$ are provided. They arise in different geometric contexts. More precisely, we construct hyperovals by means of certain subsets of the projective plane called here k-tangent arcs with respect to a Hermitian curve (Section 2), hyperovals arising from the geometry of an orthogonal polarity commuting with a unitary polarity (Section 3), hyperovals admitting the irreducible linear group $\mathrm{PSL}(2,7)$ as a subgroup of $\mathrm{PGU}(3,q^2)$, $q=p^h$, $p\equiv 3,5$ or $6 \pmod 7$ and h an odd integer (Section 4). Finally we construct hyperovals by means of the embedding of $\mathrm{PSp}(4,q) < \mathrm{PGU}(4,q^2)$ as a subfield subgroup (Section 5).

1. Introduction

A *hyperoval* or a *local subspace* of a polar space $\mathcal{P}$ is a non-empty set of points of $\mathcal{P}$ which intersects every singular line of $\mathcal{P}$ in either 0 or 2 points.

Hyperovals of polar spaces arise in the context of locally polar spaces.

Let X be a finite set with a family of subsets of cardinality at least four, called blocks. Distinct points of X are said to be adjacent if they belong to a block and the set of points adjacent to a point P is called the residual space X_P. The sets of the form $B\setminus\{P\}$, where B is a block containing P are called the lines of X_P. The set X is a locally polar space if (1) for each block B and each point P not in B, P is adjacent to 0, 2 or all points of B; (2) no point of X_P is adjacent to every other point of X_P; (3) any three pairwise adjacent points of X belong to some block. The residual spaces of a locally polar space are polar spaces. If X is connected these residual spaces have the same rank.

From a result of Buekenhout and Hubaut [2, Proposition 3] it follows that if A is a polar space of polar rank r, $r \geqslant 3$ and order n, and H a hyperoval of A, then H equipped with the graph induced by A on H, is the adjacency graph of a locally polar space of polar rank $r-1$ and order n such that the

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residual space H_P at any point $P \in H$ is isomorphic to $\mathrm{Cone}_P(A)$. This result makes the classification of all local subspaces of polar spaces interesting.

A *generalized quadrangle of order* (s,t) ($GQ(s,t)$ for short) is an incidence structure $\mathcal{S} = (P, B, I)$ of points and lines with the properties that any two points (lines) are incident with at most one line (point), every point is incident with $t+1$ lines, every line is incident with $s+1$ points, and for any point P and line l which are not incident, there is a unique point on l collinear with P. The standard reference is [15].

In this paper we will focus on the generalized quadrangle $\mathcal{H}(3,q^2)$, q odd, the incidence structure of all points and lines (generators) of a non-singular Hermitian surface in $PG(3,q^2)$, a generalized quadrangle of order (q^2,q), with automorphism group $\mathrm{P\Gamma U}(4,q^2)$; see [17] for more details.

As observed in [2, Remark 2, p. 404] when $r=2$ we can still say that a hyperoval of a generalized quadrangle $\mathcal{S}$ is a regular graph of degree equal to $|\mathrm{Cone}_P(\mathcal{S})|$, valency $t+1$, and has the property of being triangle free. More precisely, a subgraph of a point $GQ(s,t)$-graph is a hyperoval if and only if it is a regular graph of valency $t+1$ with an even number of vertices and has no triangles. A $GQ(1,t)$ has only one hyperoval: the whole point-set. For generalized quadrangles of order $(2,t)$ not many examples are known.

Lower and upper bounds on the size of a hyperoval of a generalized quadrangle of order (s,t) were obtained by Cameron, Hughes, Pasini [3], Del Fra, Ghinelli, Payne [11] and De Bruyn [10]. In particular, the following proposition appears in [10].

Proposition 1.1. *Let $\mathcal{S}$ be a generalized quadrangle of order (s,t) and let H be a hyperoval of $\mathcal{S}$. Then*

(i) *2 is a divisor of $|H|$;*
(ii) *$|H| \geqslant 2(t+1)$ and equality holds if and only if there exists a regular pair $\{x,y\}$ of non-collinear points of $\mathcal{S}$ such that $H = \{x,y\}^{\perp} \cup \{x,y\}^{\perp\perp}$;*
(iii) *$|H| \geqslant (t-s+2)(s+1)$ and if equality holds then every point outside H is incident with precisely $1 + (t-s)/2$ lines which meet H (hence $s \equiv t \pmod 2$);*
(iv) *$|H| \leqslant 2(st+1)$ and equality holds if and only if every line of $\mathcal{S}$ intersects H in exactly 2 points.*

The hyperovals of $\mathcal{H}(3,4)$ have been classified by Makhnev [13]. He found hyperovals of size 6, 8, 10, 12, 14, 16, 18.

Some infinite families of hyperovals of $\mathcal{H}(3,q^2)$, q odd, are due to Cossidente. In [4] he found two infinite families of transitive hyperovals of sizes q^3+1 and q^3-q, respectively. Other infinite families of hyperovals of $\mathcal{H}(3,q^2)$ have been constructed in [6] and [7], for q even, and in [8], for q odd. See also [16].

In this paper we will construct some new infinite families of hyperovals of $\mathcal{H}(3,q^2)$. The first construction relies on the notion of k-tangent arc with respect to a Hermitian curve which seems to be interesting in its own right. It produces hyperovals of size $2k(q+1)$, with $1 \leqslant k \leqslant q^2-q+1$, $(q^2+2)(q+1)$, when q is even, $(q^2+1)(q+1)$, when q is odd. Other two hyperovals arise from a conic disjoint from a Hermitian curve. Basing on the geometry of orthogonal polarities commuting with a non-degenerate unitary polarity [17], a slight modification of a result due to Cossidente and Ebert [5], produces hyperovals of $\mathcal{H}(3,q^2)$, q odd, of size q^3-q. The third construction is done by looking at the action of the simple group $\mathrm{PSL}(2,7)$ inside $\mathrm{PGU}(3,q^2)$, where $q=p^h$, $p \equiv 3, 5$ or $6 \pmod 7$ and h is an odd integer. Interestingly, this representation of $\mathrm{PSL}(2,7)$ is strictly related to the Coxeter graph [9] and the Klein quartic curve [18]. We obtain hyperovals of size $42(q+1)$ and $56(q+1)$, both admitting the linear group $\mathrm{PSL}(2,7)$ as an automorphism group. Finally, regarding $\mathrm{PSp}(4,q)$ as a subgroup of $\mathrm{PGU}(4,q^2)$ (subfield subgroup), stabilizing a symplectic subquadrangle $\mathcal{W}(3,q)$ of $\mathcal{H}(3,q^2)$ [15], we construct two families of hyperovals of $\mathcal{H}(3,q^2)$ of size q^2+2q, when q is even, and $(q+1)^2$, when q is odd.

2. Hyperovals arising from *k*-tangent arcs

Let $\mathcal{H}(3,q^2)$ be a Hermitian surface of $PG(3,q^2)$. Let π be a secant plane of $\mathcal{H}(3,q^2)$ meeting $\mathcal{H}(3,q^2)$ at the Hermitian curve $\mathcal{H}(2,q^2)$.