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A note on the real densities of homogeneous systems in function fields

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ABSTRACT

Let $\mathbb{F}_q((1/t))$ denote the field of formal power series in $1/t$ over the finite field $\mathbb{F}_q$ of q elements. In this paper, we prove the existence of the real densities of certain homogeneous systems in $\mathbb{F}_q((1/t))$. In addition, we show that the real density of a system is equal to the corresponding singular integral.

1. Introduction

Local densities play significant roles in asymptotic formulas for many Diophantine problems. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $k, s \in \mathbb{N} \setminus \{0\}$ and nonzero integers a_{ij} with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant s$, we write $\mathbf{x} = (x_1, \ldots, x_s)$ and

$$\phi_i(\mathbf{x}) = a_{i1}x_1^i + \cdots + a_{is}x_s^i \quad (1 \leqslant i \leqslant k). \tag{1}$$

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For $B \in \mathbb{N} \setminus \{0\}$, define $N(B)$ to be the number of integral solutions of the Diophantine system $\phi_i(\mathbf{x}) = 0$ $(1 \leqslant i \leqslant k)$ with $|x_j| \leqslant B$ $(1 \leqslant j \leqslant s)$. A result of Wooley [12, Theorem 9.1] states that when $k \geqslant 3$ and $s \geqslant 2k^2 + 2k + 1$, subject to certain local solubility conditions, one has

$$N(B) \sim C_{s,k,\mathbf{a}} B^{s - \frac{1}{2}k(k+1)},$$

where $C_{s,k,\mathbf{a}}$ is a positive constant. In particular, this constant can be factored as a product of the local densities associated with the above system defined as in Schmidt [10] and Wooley [12]. When $L > 0$, for $v \in \mathbb{R}$, define

$$\lambda_L(v) = \begin{cases} L(1 - L|v|), & \text{when } |v| \leqslant L^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

The limit

$$\mu(\infty) = \lim_{L \to \infty} \int\limits_{[-1,1]^s} \prod_{i=1}^{k} \lambda_L\big(\phi_i(\mathbf{u})\big)\, d\mathbf{u}, \tag{2}$$

when it exists, is called the *real density*. Meanwhile, given $l \in \mathbb{N} \setminus \{0\}$ and a prime number p, we write

$$\nu_l = p^{l(k-s)} \operatorname{card}\big\{\mathbf{x} \in \big(\mathbb{Z}/p^l\mathbb{Z}\big)^s \bigm| \phi_i(\mathbf{x}) \equiv 0 \ \big(\mathrm{mod}\ p^l\big)\ (1 \leqslant i \leqslant k)\big\}.$$

The limit $\nu(p) = \lim_{l \to \infty} \nu_l$, when it exists, is called the *p-adic density*. We see from [12, Theorem 9.1] that

$$C_{s,k,\mathbf{a}} = \mu(\infty)\Big(\prod_p \nu(p)\Big) \tag{3}$$

where the product is over all prime numbers. In addition, on writing $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_s)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)$, we see from Schmidt [10, Section 3] that the real density is exactly the singular integral

$$\mu(\infty) = \int\limits_{\mathbb{R}^k} \int\limits_{[-1,1]^s} e\big(\beta_1\phi_1(\boldsymbol{\gamma}) + \cdots + \beta_k\phi_k(\boldsymbol{\gamma})\big)\, d\boldsymbol{\gamma}\, d\boldsymbol{\beta}$$

where $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. Such asymptotic relations can be generalized to multiple Diophantine problems (for more details, see [9, Theorem 1.4]).

Note that the absolute value $|\cdot|$ is an archimedean valuation on the ring of integers $\mathbb{Z}$. As in [1, Introduction], we may interpret the real density $\mu(\infty)$ as the area of the manifold defined by (1) in the box $[-1,1]^s$. Let $\mathbb{F}_q[t]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ of q elements. We may also define a valuation $\langle\cdot\rangle$ on $\mathbb{F}_q[t]$ by $\langle x\rangle = q^{\deg x}$. However, this valuation is non-archimedean and such a metric cannot define a manifold. We are therefore interested in the real density of Diophantine systems over $\mathbb{F}_q[t]$. To begin with, we now set up the field $\mathbb{K}_\infty$, the completion of the fraction field of $\mathbb{F}_q[t]$ with respect to $\langle\cdot\rangle$. Indeed, $\mathbb{K}_\infty$ is equal to $\mathbb{F}_q((1/t))$, the field of formal power series in $1/t$ over $\mathbb{F}_q$. Thus, each element $\alpha \in \mathbb{K}_\infty$ can be written in the shape $\alpha = \sum_{i \leqslant n} a_i t^i$ for some $n \in \mathbb{Z}$ and coefficients $a_i = a_i(\alpha) \in \mathbb{F}_q$ $(i \leqslant n)$. We define $\operatorname{ord}\alpha$ to be the largest integer i for which $a_i(\alpha) \neq 0$ and write $\langle\alpha\rangle = q^{\operatorname{ord}\alpha}$. In this context, we adopt the convention that $\operatorname{ord} 0 = -\infty$ and $\langle 0\rangle = 0$. Let $\mathbb{T} = \{\alpha \in \mathbb{K}_\infty \mid \langle\alpha\rangle < 1\}$. We may normalize any Haar measure $d\alpha$ on $\mathbb{K}_\infty$ in such a manner that $\int_{\mathbb{T}} 1\, d\alpha = 1$.