

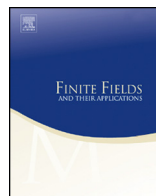


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Standard sequence subgroups in finite fields ☆

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ABSTRACT

In previous work, the authors describe certain configurations which give rise to standard and to non-standard subgroups for linear recurrences of order $k = 2$, while in subsequent work, a number of families of non-standard subgroups for recurrences of order $k \geq 2$ are described. Here we exhibit two infinite families of standard groups for $k \geq 2$.

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1. Introduction

In what follows, p will always denote a prime, q a power of p , $\mathbb{F}_q$ the field of order q and $\mathbb{A}_q$ a fixed algebraic closure of $\mathbb{F}_q$. We will assume that all our finite extensions of $\mathbb{F}_q$ are subfields of $\mathbb{A}_q$. Further, k will be a positive integer and $\mathbb{N}$ will denote the set of all positive integers.

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Definition 1.1. Let

$$f(t) = t^k - f_{k-1}t^{k-1} - \cdots - f_1t - f_0 \in \mathbb{F}_q[t]$$

where $f(0) \neq 0$.

(a) An f -sequence in $\mathbb{A}_q$ is a (doubly-infinite) sequence $S = (s_i)_{i \in \mathbb{Z}}$ of elements $s_i \in \mathbb{A}_q$ such that

$$s_i = f_{k-1}s_{i-1} + \cdots + f_1s_{i-k+1} + f_0s_{i-k}$$

for all $i \in \mathbb{Z}$.

(b) An f -subgroup is a finite subgroup $M \leq \mathbb{L}^*$, where $\mathbb{L} \subseteq \mathbb{A}_q$ is a finite extension of $\mathbb{F}_q$, such that M may be written as (the underlying set of a minimal periodic segment of) a periodic f -sequence

$$(\cdots, m_0 = 1, m_1, \dots, m_{|M|-1}, \dots)$$

of least period $|M|$, where $|M|$ denotes the order of M . In this situation we say that the f -sequence $(m_i)_{i \in \mathbb{Z}}$ represents M as an f -subgroup.

(c) The f -sequence $S = (s_i)_{i \in \mathbb{Z}}$ in $\mathbb{A}_q^*$ is called *cyclic* if there exists $\lambda \in \mathbb{A}_q^*$ such that $s_{i+1} = \lambda s_i$ for all $i \in \mathbb{Z}$; in this situation, λ will be called the *common ratio* of S .

(d) The *unit f -sequence*, $\mathcal{U} = (u_n)_{n \in \mathbb{Z}}$, is the f -sequence in $\mathbb{F}_q$ defined by $u_0 = \cdots = u_{k-2} = 0$, $u_{k-1} = 1$ if $k > 1$; when $k = 1$ the unit f -sequence will be the f -sequence defined by $u_0 = 1$.

(e) The *restricted period*, $\delta(f)$ of f , is defined to be 1 if $k = 1$ and is the least integer $n > 0$ with $u_n = \cdots = u_{n+k-2} = 0$ if $k > 1$ (see [3]).

In (a) it is known (because $f(0) \neq 0$) that an f -sequence must be periodic: see 8.11 of [8]. In (e), it is clear that if $k > 1$ then $\delta(f) \geq k$.

The following lemma relates f -subgroups with cyclic f -sequences.

Lemma 1.2. Suppose that $f(t) \in \mathbb{F}_q[t]$ is monic of degree k with $f(0) \neq 0$.

(a) Suppose that S is a non-null cyclic f -sequence in $\mathbb{A}_q^*$ with common ratio $\lambda \neq 0$. If S contains 1 then S represents $\langle \lambda \rangle \leq \mathbb{A}^*$ as an f -subgroup and $f(\lambda) = 0$.

(b) Let M be an f -subgroup. Then M is a cyclic group. If S is a cyclic f -sequence which represents M then the common ratio λ of S satisfies $M = \langle \lambda \rangle$ and $f(\lambda) = 0$.

(c) Suppose $M \leq \mathbb{A}_q^*$ is finite. Suppose $M = \langle \lambda \rangle$ and let $m(t)$ be the minimum polynomial of λ over $\mathbb{F}_q$. Then M is an m -group and also an f -group for any multiple $f(t)$ of $m(t)$ in $\mathbb{F}_q[t]$.

Proof. For (a) and (b), see Lemma 1.3 of [5]. Note that in (a), S is periodic because $f(0) \neq 0$ and so λ has finite multiplicative order, while in (b) a finite subgroup of the multiplicative group of a field is always cyclic: see Exercise 2.9 in [8].

(c) It is clear that $\{1, \lambda, \dots\}$ exhibits $M = \langle \lambda \rangle$ as an m -sequence; then by Theorem 8.42 of [8], M is an f -subgroup for any multiple $f(t)$ of $m(t)$ in $\mathbb{F}_q[t]$. $\square$

The motivation for studying f -subgroups seems to go back to Somer [9,10]. In particular, if $\omega \in \mathbb{A}_q^*$ is a root of $f(t) \in \mathbb{F}_q[t]$ then $\langle \omega \rangle \leq \mathbb{A}_q^*$ may be regarded as (the underlying set of) an f -sequence of minimal period $|\omega|$:

$$\langle \omega \rangle = (\cdots, 1, \omega, \omega^2, \dots, \omega^{|\omega|-1}, \dots).$$

It can sometimes happen, for certain choices of $\mathbb{F}_q$, $f(t)$ and ω with $f(\omega) = 0$, that the subgroup $\langle \omega \rangle$ may be represented in an alternative, “less obvious”, manner as an f -sequence; this leads to the following definition:

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