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The asymptotic couple of the field of logarithmic transseries

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ABSTRACT

The derivation on the differential-valued field $\mathbb{T}_{\log}$ of logarithmic transseries induces on its value group $\Gamma_{\log}$ a certain map ψ. The structure $(\Gamma_{\log}, \psi)$ is a divisible asymptotic couple. We prove that the theory $T_{\log} = \mathrm{Th}(\Gamma_{\log}, \psi)$ admits elimination of quantifiers in a natural first-order language. All models (Γ, ψ) of $T_{\log}$ have an important discrete subset $\Psi := \psi(\Gamma \setminus \{0\})$. We give explicit descriptions of all definable functions on Ψ and prove that Ψ is stably embedded in Γ.

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1. Introduction

The differential-valued field $\mathbb{T}_{\log}$ of logarithmic transseries is conjectured to have good model theoretic properties. As a partial result in this direction, and as a confidence building measure we prove here that at least its *asymptotic couple* has a good model theory: quantifier elimination, and stable embeddedness of a certain discrete part. We now describe the relevant objects and results in more detail.

Throughout, m and n range over $\mathbb{N} = \{0, 1, 2, \ldots\}$. See [5] for a definition of the differential-valued field $\mathbb{T}_{\log}$ of logarithmic transseries. It is a field extension of $\mathbb{R}$ containing elements $\ell_0, \ell_1, \ell_2, \ldots$, to be thought of as $x, \log x, \log\log x, \ldots$, and the elements of $\mathbb{T}_{\log}$ are formal series with real coefficients and monomials $\ell_0^{r_0}\ell_1^{r_1}\cdots\ell_n^{r_n}$ (with arbitrary real exponents $r_0, \ldots, r_n$). For our purpose it is enough to know the following four things about $\mathbb{T}_{\log}$, its elements ℓ_n, and these monomials:

1. These monomials are the elements of a subgroup $\mathfrak{L}$ of the multiplicative group of $\mathbb{T}_{\log}$, and their products are formed in the way suggested by their notation as power products. The elements of $\mathfrak{L}$ are also known as *logarithmic monomials*. For $m \leq n$ we have $\ell_m = \ell_0^{r_0}\cdots\ell_n^{r_n}$ where $r_i = 0$ for all $i \neq m$ and $r_m = 1$.
2. The field $\mathbb{T}_{\log}$ is equipped with a (Krull) valuation v that maps the group $\mathfrak{L}$ isomorphically onto the (additively written) value group $v(\mathbb{T}_{\log}^{\times}) = \bigoplus_n \mathbb{R}e_n$, a vector space over $\mathbb{R}$ with basis (e_n), with
$$v(\ell_0^{r_0}\ell_1^{r_1}\cdots\ell_n^{r_n}) \;=\; -\,r_0e_0 - \cdots - r_ne_n,$$
and made into an ordered group by requiring for nonzero $\sum_i r_ie_i$ that
$$\sum r_ie_i > 0 \iff r_n > 0 \;\text{ for the least } n \text{ such that } r_n \neq 0.$$
3. The field $\mathbb{T}_{\log}$ is equipped with a derivation such that $\ell_0' = 1, \ell_1' = \ell_0^{-1}$, and in general $\ell_n^{\dagger} = \ell_0^{-1}\cdots\ell_n^{-1}$. Here $f^{\dagger} := f'/f$ denotes the logarithmic derivative of a nonzero element f of a differential field, obeying the useful identity $(fg)^{\dagger} = f^{\dagger} + g^{\dagger}$. In $\mathbb{T}_{\log}$,
$$(\ell_0^{r_0}\ell_1^{r_1}\cdots\ell_n^{r_n})^{\dagger} \;=\; r_0\ell_0^{-1} + r_1\ell_0^{-1}\ell_1^{-1} + \cdots + r_n\ell_0^{-1}\cdots\ell_n^{-1}.$$
4. This derivation has the property that for nonzero $f \in \mathbb{T}_{\log}$ with $v(f) \neq 0$, the value $v(f')$, and thus $v(f^{\dagger})$, depends only on $v(f)$.

Let $\Gamma_{\log}$ be the above ordered abelian group $\bigoplus_n \mathbb{R}e_n$. For an arbitrary ordered abelian group Γ we set $\Gamma^{\neq} := \Gamma \setminus \{0\}$. By (4) the derivation of $\mathbb{T}_{\log}$ induces maps

$$\gamma \mapsto \gamma' \;\text{ and }\; \gamma \mapsto \gamma^{\dagger} \;:\; \Gamma_{\log}^{\neq} \to \Gamma_{\log}$$