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# On the exactness of products in the localization of (Ab.4*) Grothendieck categories

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ABSTRACT

Let $\mathfrak{C}$ be an (Ab.4*) Grothendieck category, that is, products are exact in $\mathfrak{C}$. Given a hereditary torsion class $\mathcal{T} \subseteq \mathfrak{C}$, we study the exactness of products in the Gabriel localization $\mathfrak{C}/\mathcal{T}$ of $\mathfrak{C}$. We show that, under suitable assumptions on $\mathfrak{C}$, the $k+1$-th derived functor of the product vanishes, provided the Gabriel dimension of $\mathfrak{C}/\mathcal{T}$ is smaller than $k$. As a consequence, we deduce that, under suitable hypotheses, the derived category $\mathbf{D}(\mathfrak{C}/\mathcal{T})$ is left-complete.



## 1. Introduction

Let $\mathfrak{C}$ be a Grothendieck category. Remember from [27] that, given $n \in \mathbb{N}$, $\mathfrak{C}$ is said to be (Ab.4*)-$n$, if the $k$-th derived functor of the product vanishes for all $k > n$ (see

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Definition 3.8). In particular, products are exact in $\mathfrak{C}$ if and only if $\mathfrak{C}$ is (Ab.4*)-0, that is, $\mathfrak{C}$ satisfies Grothendieck's (Ab.4*) condition.

In this paper we study the following question:

Given an (Ab.4*) Grothendieck category $\mathfrak{C}$ and a hereditary torsion class $\mathcal{T} \subseteq \mathfrak{C}$, what can be said about the exactness of products in the Gabriel localization $\mathfrak{C}/\mathcal{T}$?

To answer the above question we proceed by steps. First of all we recall in Section 2 the needed background about Grothendieck categories, torsion theories, localization and (relative) Gabriel dimension. Furthermore, we introduce in Subsection 2.4 the concept of an *effective Grothendieck category*, that is a locally Noetherian, stable Grothendieck category in which all the prime torsion theories are exact. A natural example of an effective Grothendieck category is the category of modules over a commutative Noetherian ring.

Let now $\mathfrak{C}$ be a Grothendieck category, let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory (see Definition 2.1), and denote by $\mathbf{T}_\tau \colon \mathfrak{C} \to \mathcal{T}$ the torsion functor (see Definition 2.5). In Section 3 we define, for any $k \in \mathbb{N}$, the $k$-th $\tau$-*local cohomology* functor

$$\Gamma^k_\tau \colon \mathfrak{C} \to \mathcal{T}$$

as the $k$-th right derived functor of $\mathbf{T}_\tau$. If $\mathfrak{C}$ is effective, we can prove that $\Gamma^k_\tau(X) = 0$ for all the objects $X \in \mathfrak{C}$ whose image in $\mathfrak{C}/\mathcal{T}$ has suitably small Gabriel dimension (see Theorem 3.7 for details).

This vanishing result is applied in Theorem 3.10 to show that, if $\mathfrak{C}$ is an (Ab.4*) effective Grothendieck category and if the Gabriel dimension $\mathrm{G.dim}(\mathfrak{C}/\mathcal{T}) = k < \infty$, then $\mathfrak{C}/\mathcal{T}$ is (Ab.4*)-$k+1$. In this way we obtain a partial answer to the above general question.

Finally in Subsection 3.3 we use a recent result from [19] to deduce that, under the above hypotheses, the derived category $\mathbf{D}(\mathfrak{C}/\mathcal{T})$ is left-complete, that is, any unbounded complex $X^\bullet \in \mathbf{D}(\mathfrak{C}/\mathcal{T})$ is quasi-isomorphic to the homotopy limit of its left truncations (see Definition 3.12). We also discuss how this fact can be used to give an "explicit" construction of DG-injective resolutions of unbounded complexes.

In the last two sections of the paper we apply the above theory to some problems investigated in [7,5,6]. Indeed, given a Grothendieck category $\mathfrak{C}$, remember from [7] that an *injective class of injectives* is a class $\mathcal{I}$ of injective objects, closed under summands and arbitrary products, and such that any object $X \in \mathfrak{C}$ admits a morphism $\phi \colon X \to I$ with $I \in \mathcal{I}$ and such that $\mathrm{Hom}_{\mathfrak{C}}(\phi, J)$ is surjective for all $J \in \mathcal{I}$.

Our first observation in Section 4 is that injective classes of injectives correspond exactly to the hereditary torsion theories in $\mathfrak{C}$ (see Theorem 4.8). This fact, together with some classical results about hereditary torsion theories, allows us to re-obtain and generalize the main results of [7] classifying the injective classes of injectives in categories of modules (see Corollaries 4.12 and 4.15).