Journal of Algebra 470 (2017) 68–76

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Direct limits and inverse limits of Mackey functors

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A R T I C L E   I N F O

Article history:
Received 16 December 2015
Available online 9 September 2016
Communicated by Michel Broué

MSC:
primary 19A22
secondary 57S17

Keywords:
Mackey functor
Green ring functor
Burnside ring

A B S T R A C T

Let G be a finite group, $\mathbb{S}$ the subgroup category of G, and M a Mackey functor on G. For full subcategories $\mathbb{G}$ and $\mathbb{H}$ of $\mathbb{S}$, we have the direct limit $M_*(\mathbb{G})$ and the inverse limit $M^*(\mathbb{H})$ of M. In this paper we study relation between the canonical homomorphisms $\mathrm{ind} : M_*(\mathbb{G}) \to M(G)$ and $\mathrm{res} : M(G) \to M^*(\mathbb{H})$.

1. Introduction

Throughout this paper G is a finite group. Let $\mathcal{S} = \mathcal{S}(G)$ denote the set of all subgroups of G. For a category $\mathbb{K}$, we denote by $\mathrm{Obj}(\mathbb{K})$ the totality of objects in $\mathbb{K}$, and by $\mathrm{Mor}_{\mathbb{K}}(x, y)$ the set of morphisms from x to y in $\mathbb{K}$, where x, $y \in \mathrm{Obj}(\mathbb{K})$, and by $\mathrm{Mor}(\mathbb{K})$ the totality $\coprod_{x,y\in\mathrm{Obj}(\mathbb{K})} \mathrm{Mor}_{\mathbb{K}}(x, y)$. Let $\mathbb{S}$ denote the subgroup category of G of which the objects are all subgroups of G, i.e. $\mathrm{Obj}(\mathbb{S}) = \mathcal{S}$, of which the morphisms from $H \in \mathcal{S}$ to $K \in \mathcal{S}$ are all triples (H, g, K) consisting of $g \in G$ with ${}^gH \subset K$, where ${}^gH = gHg^{-1}$, and in which the composition $(K, y, L) \circ (H, x, K)$ of morphisms (H, x, K) and (K, y, L) is given by (H, yx, L).

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http://dx.doi.org/10.1016/j.jalgebra.2016.09.002

Let $\mathbb{A}$ denote the category of abelian groups whose morphisms are all group homomorphisms. Throughout this paper, let $M = (M_*, M^*) : \mathbb{S} \to \mathbb{A}$ be a Mackey functor on G in the sense of A. Bak [2], where M_* and M^* are covariant and contravariant functors, respectively, and $M_*(H)$ coincides with $M^*(H)$, and both are usually denoted by $M(H)$ for $H \in \mathcal{S}$.

Let $\mathbb{B}^w$ denote the category of associative rings with unity whose morphisms are all ring homomorphisms preserving unity between objects, and whose weak morphisms are all group homomorphisms between objects. Let $F = (F_*, F^*) : \mathbb{S} \to \mathbb{B}^w$ be a Green ring functor on G in the sense of A. Bak [2]. Thus F is a bifunctor, $F^*(f)$ and $F_*(f)$ are a morphism and a weak morphism in $\mathbb{B}^w$, respectively, for each $f \in \mathrm{Mor}(\mathbb{S})$, and F satisfies the Mackey subgroup property and the Frobenius reciprocity law. For $H \in \mathcal{S}$, let 1_H denote the unity of the ring $F(H)$ $(=F^*(H))$.

We recall that any Mackey functor on G is a Green module over the Burnside ring functor $A = (A_*, A^*) : \mathbb{S} \to \mathbb{B}^w$ of G, i.e. $A(H)$ is the Grothendieck group of the category of finite H-sets for $H \in \mathcal{S}$, cf. [5,3,4]. As usual, $M_*(f)$ and $F_*(f)$ (resp. $M^*(f)$ and $F^*(f)$) are denoted by f_* (resp. f^*) for $f \in \mathrm{Mor}(\mathbb{S})$, and $(H, e, K)_*$, $(H, e, K)^*$, $(H, g, {}^gH)_*$, and $(H, g, {}^gH)^*$ are denoted by ind_H^K, res_H^K, $c(g)_*$, and $c(g)^*$, respectively, where e is the identity element of G.

Let $\mathcal{G}$ be a subset of $\mathcal{S}$. We call $\mathcal{G}$ *lower closed* if $\mathcal{S}(H) \subset \mathcal{G}$ holds for all $H \in \mathcal{G}$. We call $\mathcal{G}$ *conjugation invariant* if ${}^gH \in \mathcal{G}$ holds for all $H \in \mathcal{G}$ and $g \in G$. In the following, let $\mathcal{G}$ and $\mathcal{H}$ be lower closed and conjugation invariant subsets of $\mathcal{S}$. Let $\mathfrak{S}(G, \mathcal{G})$ denote the full subcategory of $\mathbb{S}$ such that $\mathrm{Obj}(\mathfrak{S}(G, \mathcal{G})) = \mathcal{G}$ and set $\mathbb{G} = \mathfrak{S}(G, \mathcal{G})$. We similarly define $\mathbb{H}$ for $\mathcal{H}$, i.e. $\mathbb{H} = \mathfrak{S}(G, \mathcal{H})$. The direct limit $M_*(\mathbb{G})$ and the inverse limit $M^*(\mathbb{G})$ of M for $\mathbb{G}$ are defined by

$$M_*(\mathbb{G}) = \left(\bigoplus_{H \in \mathcal{G}} M(H) \right) \Bigg/ \left\langle x - y \;\middle|\; \begin{array}{c} x \in M(H),\, y \in M(K),\, f_* x = y \\ \text{for some } f \in \mathrm{Mor}_{\mathbb{G}}(H, K) \end{array} \right\rangle \text{ and}$$

$$M^*(\mathbb{G}) = \left\{ (x_H) \in \prod_{H \in \mathcal{G}} M(H) \;\middle|\; \begin{array}{c} x_H = f^* x_K \text{ for all} \\ H,\, K \in \mathcal{G},\, f \in \mathrm{Mor}_{\mathbb{G}}(H, K) \end{array} \right\},$$

cf. [1, p. 243]. The induction homomorphism $\mathrm{ind}_{\mathbb{G}}^G : M_*(\mathbb{G}) \to M(G)$ and the restriction homomorphism $\mathrm{res}_{\mathbb{H}}^G : M(G) \to M^*(\mathbb{H})$ are canonically given. Let $\psi_{\mathbb{G},\mathbb{H}}^G$ denote the composition $M_*(\mathbb{G}) \to M^*(\mathbb{H})$ of $\mathrm{ind}_{\mathbb{G}}^G$ and $\mathrm{res}_{\mathbb{H}}^G$.

Let $\mathcal{H}_G = \mathcal{S}(G) \smallsetminus \{G\}$ and $\mathbb{H}_G = \mathfrak{S}(G, \mathcal{H}_G)$. We remark that for any nontrivial group G, the cokernel of $\mathrm{ind}_{\mathbb{H}_G}^G : A_*(\mathbb{H}_G) \otimes \mathbb{Q} \to A(G) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-module with basis $\{1_G = [G/G]\}$, where $\mathbb{Q}$ is the field of rational numbers. Yasuhiro Hara found that for the cartesian product $G = C_{p_1} \times \cdots \times C_{p_m}$ of cyclic groups C_{p_i} of order p_i, where each p_i is a prime, the restriction homomorphism $\mathrm{res}_{\mathbb{H}_G}^G : A(G) \to A^*(\mathbb{H}_G)$ is surjective if and only if the primes $p_1, \ldots, p_m$ are mutually distinct, i.e. G is cyclic. Masafumi Sugimura remarked that $\psi_{\mathbb{H}_G,\mathbb{H}_G}^G : A_*(\mathbb{H}_G) \to A^*(\mathbb{H}_G)$ is surjective for $G = A_5$ the alternating group on five