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On syzygies over 2-Calabi–Yau tilted algebras[☆]

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ABSTRACT

We characterize the syzygies and co-syzygies over 2-Calabi–Yau tilted algebras in terms of the Auslander–Reiten translation and the syzygy functor. We explore connections between the category of syzygies, the category of Cohen–Macaulay modules, the representation dimension of algebras and the Igusa–Todorov functions. In particular, we prove that the Igusa–Todorov dimensions of d-Gorenstein algebras are equal to d.

For cluster-tilted algebras of Dynkin type $\mathbb{D}$, we give a geometric description of the stable Cohen–Macaulay category in terms of tagged arcs in the punctured disc. We also describe the action of the syzygy functor in a geometric way. This description allows us to compute the Auslander–Reiten quiver of the stable Cohen–Macaulay category using tagged arcs and geometric moves.

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1. Introduction

The concept of a 2-Calabi–Yau tilted algebra is a natural generalization of the concept of a cluster-tilted algebra. A cluster-tilted algebra is the endomorphism algebra of a cluster-tilting object in the cluster category of a hereditary algebra. The 2-Calabi–Yau tilted algebras are obtained by replacing the cluster category by a 2-Calabi–Yau triangulated category. Examples of 2-Calabi–Yau tilted algebras that are not cluster-tilted are the Jacobian algebras of the quivers with potential arising from a cluster algebra of non-acyclic type. In particular, this includes the Jacobian algebras from triangulated surfaces other than the disc with 0, 1, or 2 punctures and the annulus without punctures. Cluster categories and cluster-tilted algebras were introduced in [11,14,12]. The generalization to 2-Calabi–Yau categories was given in [24]. Cluster algebras were defined in [19] and their quivers with potentials in [17]. For triangulated surfaces, the Jacobian algebras were introduced in [3,25].

From now on, let B be a 2-Calabi–Yau tilted algebra. We study the stable category $\underline{\mathrm{CM}}(\mathrm{B})$ of Cohen–Macaulay modules over B. A B-module M is (maximal) Cohen–Macaulay if $\mathrm{Ext}^i_{\mathrm{B}}(M, \mathrm{B}) = 0$, for all $i > 0$. It was proved in [24] that B is d-Gorenstein, with $d = 0$ or 1, and that the stable category of Cohen–Macaulay modules is 3-Calabi–Yau. We denote by $\mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$ the set of indecomposable objects in $\underline{\mathrm{CM}}(\mathrm{B})$. Let Ω be the syzygy functor and τ the Auslander–Reiten translation. Our first main result gives the following characterization.

Theorem 1. *Let M be an indecomposable* B*-module. Then the following statements are equivalent.*

(1) *M is a non-projective syzygy,*
(2) $M \in \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$,
(3) $\Omega^2 \tau M \simeq M$,
(4) *M is non-projective and $\Omega^{-2} M \simeq \tau M$.*

We obtain the following corollary on selfinjective 2-Calabi–Yau tilted algebras.

Corollary 1. *Suppose that* B *is selfinjective and such that the Nakayama functor has finite order m. Let M be an indecomposable, non-projective* B*-module. Then $\tau^{2m} M \cong M$. In particular, if* B *is symmetric then $\tau^2 M \cong M$.*

The representation dimension of cluster-tilted algebras has been studied in [20] for the special case of cluster concealed algebra, thus cluster-tilted algebras whose corresponding cluster-tilting object is preprojective. The authors of [20] have shown that in this case the representation dimension is at most 3. Cluster concealed algebras may or may not be tame and, on the other hand, tame cluster-tilted algebras need not be cluster concealed. Using results of [9] and [29], we extend the result to tame cluster-tilted algebras.