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On the Frattini subgroup of a finite group ☆

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A B S T R A C T

We study the class of finite groups G satisfying $\Phi(G/N) = \Phi(G)N/N$ for all normal subgroups N of G. As a consequence of our main results we extend and amplify a theorem of Doerk concerning this class from the soluble universe to all finite groups and answer in the affirmative a long-standing question of Christensen whether the class of finite groups which possess complements for each of their normal subgroups is subnormally closed.

1. Introduction and statement of results

The only groups considered in this paper are finite. In the present article we shall examine certain questions concerning the behaviour of the Frattini subgroup in epimorphic images.

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Following Gaschütz [10], we call a group Φ-free if its Frattini subgroup is trivial. We denote by $\mathfrak{B}$ the class of groups G such that G/N is Φ-free for all normal subgroups N of G. It is clear that a group $G \in \mathfrak{B}$ if and only if G has no Frattini chief factors (recall that a chief factor K/L of a group G is said to be Frattini if $\Phi(G/L) \geq K/L$).

Our first main result is a reduction theorem of wide applicability that provides a sufficient condition for a Φ-free group to belong to $\mathfrak{B}$.

Theorem A. *Suppose that G is a Φ-free group. If G/N is Φ-free for all normal subgroups N of G containing the generalised Fitting subgroup $\mathrm{F}^*(G)$ then G belongs to $\mathfrak{B}$.*

Theorem A is important in the study of groups that behave like nilpotent groups with respect to the Frattini subgroup, that is, $\Phi(G/N) = \Phi(G)N/N$ for all $N \lhd G$. In one of his last papers, Doerk [9] examined the soluble case and proved that the class $\mathfrak{F}_{\mathrm{sol}}$ of soluble groups G where $\Phi(G/N) = \Phi(G)N/N$ for every normal subgroup N of G is a saturated formation, that is, a class of groups which is closed under epimorphic images, subdirect products and Frattini extensions. Further, he obtained several equivalent conditions for a soluble group to be in $\mathfrak{F}_{\mathrm{sol}}$.

Theorem 1.1 *([9, Satz 2']). Let G be a soluble group. Then the following statements are pairwise equivalent:*

(1) $G \in \mathfrak{F}_{\mathrm{sol}}$.
(2) *$G/\Phi(G)$ has no Frattini chief factors.*
(3) *$G/\mathrm{F}(G)$ has no Frattini chief factors.*
(4) *If H/K is a chief factor of G then $G/C_G(H/K)$ has no Frattini chief factors.*

One well-known feature of a saturated formation $\mathfrak{F}$ is that in each group G, every chief factor of the form $G^{\mathfrak{F}}/K$ is supplemented in G, where $G^{\mathfrak{F}}$ is the $\mathfrak{F}$-residual of G, that is, the smallest normal subgroup of G with quotient in $\mathfrak{F}$. *Totally non-saturated formations* (or tn-formations for short) are studied in [8,5] in the soluble universe, and in [1,3] in the general case, as the formations $\mathfrak{F}$ such that, in each group G, every chief factor of the form $G^{\mathfrak{F}}/K$ is Frattini.

Totally non-saturated formations are significant in many ways, one of which is the fact that every soluble group can be embedded in a multiprimitive group ([12]), and these groups belong to every totally non-saturated formation of full characteristic.

Our second main result is a generalisation of Doerk's theorem, valid for all groups. Its proof depends heavily on Theorem A.

Theorem B. *The following assertions are valid:*

(1) *The class $\mathfrak{B}$ is a subnormally closed tn-formation.*
(2) *The class $\mathfrak{F}$ of all groups G such that $\Phi(G/N) = \Phi(G)N/N$ for all $N \lhd G$ satisfies $\mathfrak{F} = E_\Phi\, \mathfrak{B} = \mathfrak{N}\,\mathfrak{B}$, where $\mathfrak{N}$ is the class of all nilpotent groups.*