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Huppert's Conjecture for alternating groups

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ABSTRACT

We prove that the alternating groups of degree at least 5 are uniquely determined up to an abelian direct factor by the set of degrees of their irreducible complex representations. This confirms Huppert's Conjecture for alternating groups.

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1. Introduction

Let G be a finite group. Denote by $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_k\}$ the set of all complex irreducible characters of G. Let $\mathrm{cd}(G)$ be the set of all irreducible character degrees of G forgetting multiplicities, that is,

$$\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}.$$

It is well known that the complex group algebra $\mathbb{C}G$ of G admits a decomposition

$$\mathbb{C}G = \mathrm{M}_{n_1}(\mathbb{C}) \oplus \mathrm{M}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{n_k}(\mathbb{C}),$$

where $n_i := \chi_i(1)$, for $1 \leq i \leq k$. Therefore, the complex group algebra $\mathbb{C}G$ determines the character degrees of G and their multiplicities.

An important question in character theory is whether one can recover a group or its properties from its character degrees with or without multiplicity. In other words, how much does $\mathbb{C}G$ or $\mathrm{cd}(G)$ know about the structure of G?

In general, the complex group algebras and hence the character degree sets do not uniquely determine the groups. For example, the dihedral group D_8 and the quaternion group Q_8, both of order 8, have the same character table and thus their complex group algebras are isomorphic but the groups are not isomorphic. We also have that $\mathrm{cd}(\mathrm{D}_8) = \mathrm{cd}(\mathrm{S}_3) = \{1, 2\}$. Hence the character degree sets cannot recognize nilpotency; however, the complex group algebras can (see Isaacs [11]).

Recently, G. Navarro [15] showed that the character degree set alone cannot determine the solvability of the group. Indeed, he constructed a finite perfect group H and a finite solvable group G such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. More surprisingly, Navarro and Rizo [16] found a finite perfect group and a finite nilpotent group with the same character degree set. Notice that in both examples, these finite perfect groups are not nonabelian simple. It remains open whether the complex group algebra can determine the solvability of the group or not. This is related to Brauer's Problem 2 [4], which asks when nonisomorphic groups have isomorphic group algebras.

For nonabelian simple groups and related groups, the situation is much different as pointed out in [9]. Indeed, it has been proved recently that all quasisimple groups are uniquely determined up to isomorphism by their complex group algebras. (See [2].) Recall that a finite group G is *quasisimple* if G is perfect and $G/\mathbf{Z}(G)$ is a nonabelian simple group. It turns out that a stronger result might hold for nonabelian simple group as proposed by B. Huppert [9] in the following conjecture.

Huppert's Conjecture. *Let H be any finite nonabelian simple group and G be a finite group such that* $\mathrm{cd}(G) = \mathrm{cd}(H)$*. Then $G \cong H \times A$, where A is abelian.*

Notice that Huppert's Conjecture is best possible in the sense that if $G = H \times A$ with A abelian, then $\mathrm{cd}(G) = \mathrm{cd}(H)$. In this paper, we prove the following result.