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Three dimensional Sklyanin algebras and Gröbner bases

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A B S T R A C T

We consider a Sklyanin algebra S with 3 generators, which is the quadratic algebra over a field $\mathbb{K}$ with 3 generators x, y, z given by 3 relations $pxy + qyx + rzz = 0$, $pyz + qzy + rxx = 0$ and $pzx + qxz + ryy = 0$, where $p, q, r \in \mathbb{K}$. This class of algebras enjoyed much of attention, in particular, using tools from algebraic geometry, Feigin, Odesskii [15], and Artin, Tate and Van den Bergh [3], showed that if at least two of the parameters p, q and r are non-zero and at least two of three numbers p^3, q^3 and r^3 are distinct, then S is Koszul and has the same Hilbert series as the algebra of commutative polynomials in 3 variables.

It became commonly accepted, that it is impossible to achieve the same objective by purely algebraic and combinatorial means, like the Gröbner basis technique. The main purpose of this paper is to trace the combinatorial meaning of the properties of Sklyanin algebras, such as Koszulity, PBW, PHS, Calabi–Yau, and to give a new constructive proof of the above facts due to Artin, Tate and Van den Bergh.

Further, we study a wider class of Sklyanin algebras, namely the situation when all parameters of relations could be different. We call them generalized Sklyanin algebras. We classify up to isomorphism all generalized Sklyanin algebras with the same Hilbert series as commutative polynomials on 3 variables. We show that generalized Sklyanin algebras in

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general position have a Golod–Shafarevich Hilbert series (with exception of the case of field with two elements).

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1. Introduction

It is well-known that algebras arising in string theory, from the geometry of Calabi–Yau manifolds, that is, various versions of Calabi–Yau algebras, enjoy the potentiality-like properties. This in essence comes from the symplectic structure on the manifold. The notion of *noncommutative potential* was first introduced by Kontsevich in [13]. Let $F = \mathbb{C}\langle x_1, \dots, x_n\rangle$, then the quotient vector space $F_{cyc} = F/[F, F]$ has a simple basis labeled by cyclic words in the alphabet $x_1, \dots, x_n$. For each $j = 1, \dots, n$ in [13] it was introduced a linear map $\frac{\delta}{\delta x_j} : F_{cyc} \to F$ defined by its action on monomials $\Phi = x_{i_1} \dots x_{i_n}$ by

$$\frac{\delta \Phi}{\delta x_j} = \sum_{s|i_s=j} x_{i_s+1} x_{i_s+2} \dots x_{i_r} x_{i_1} x_{i_2} \dots x_{i_s-1}$$

So, for any element $\Phi \in F_{cyc}$, which is called a potential, one can define a collection of elements $\frac{\delta\Phi}{\delta x_i}$ for $1 \leqslant i \leqslant n$. An algebra which has a presentation:

$$\mathcal{U} = \mathbb{C}\langle x_1, \dots, x_n\rangle / \left\{ \frac{\delta\Phi}{\delta x_i} \right\}_{1 \leqslant i \leqslant n}$$