Journal of Algebra 470 (2017) 441–449

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Brauer indecomposability of Scott modules

Hiroki Ishioka *, Naoko Kunugi

Department of Mathematics, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

ARTICLE INFO

Article history:
Received 24 February 2015
Available online 29 September 2016
Communicated by Michel Broué

Keywords:
Representations of finite groups
Fusion systems
Scott modules

ABSTRACT

Let k be an algebraically closed field of prime characteristic p, G a finite group and P a p-subgroup of G. We investigate the relationship between the fusion system $\mathcal{F}_P(G)$ and the Brauer indecomposability of the Scott kG-module in the case that P is not necessarily abelian. We give an equivalent condition for Scott kG-module with vertex P to be Brauer indecomposable.

1. Introduction

Let G be a finite group and k an algebraically closed field of prime characteristic p. For a p-subgroup Q of G and a finite dimensional kG-module M, the Brauer quotient $M(Q)$ of M with respect to Q has a natural structure of a $kN_G(Q)$-module. A kG-module M is said to be Brauer indecomposable if $M(Q)$ is indecomposable or zero as a $kQC_G(Q)$-module for any p-subgroup Q of G [6]. Brauer indecomposability of p-permutation modules is important for constructing stable equivalences of Morita type between blocks of finite groups (see [2]).

For subgroups Q, R of G, we denote by $\mathrm{Hom}_G(Q, R)$ the set of all group homomorphisms from Q to R which are induced by conjugation by some element of G. For a

* Corresponding author.
E-mail addresses: 1114701@ed.tus.ac.jp (H. Ishioka), kunugi@rs.kagu.tus.ac.jp (N. Kunugi).

http://dx.doi.org/10.1016/j.jalgebra.2016.09.021

p-subgroup P of G, the fusion system $\mathcal{F}_P(G)$ of G over P is the category whose objects are the subgroups of P and whose morphism set from Q to R is $\mathrm{Hom}_G(Q, R)$. We refer the reader to [1] for background involving fusion systems.

A relationship between Brauer indecomposability of p-permutation kG-modules and fusion systems was given in [6]. The main result in [6] is the following.

Theorem 1.1 *([6, Theorem 1.1]). Let P be a p-subgroup of G and M an indecomposable p-permutation kG-module with vertex P. If M is Brauer indecomposable, then $\mathcal{F}_P(G)$ is a saturated fusion system.*

In the special case that P is abelian and M is the Scott kG-module $S(G, P)$, the converse of the above theorem holds.

Theorem 1.2 *([6, Theorem 1.2]). Let P be an abelian p-subgroup of G. If $\mathcal{F}_P(G)$ is saturated, then $S(G, P)$ is Brauer indecomposable.*

We do not know whether the above theorem holds for non-abelian P. However, there are some cases in which the Scott kG-module $S(G, P)$ is Brauer indecomposable, even if P is not necessarily abelian (see [10]).

We investigate the condition that $S(G, P)$ to be Brauer indecomposable where P is not necessarily abelian. The following result is one of the main results of this paper.

Theorem 1.3. *Let G be a finite group and P a p-subgroup of G. Suppose that $M = S(G, P)$ and that $\mathcal{F}_P(G)$ is saturated. Then the following are equivalent.*

(i) *M is Brauer indecomposable.*
(ii) $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ *is indecomposable for each fully normalized subgroup Q of P.*

If these conditions are satisfied, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for each fully normalized subgroup $Q \leq P$.

A similar result is obtained independently in [5] by R. Kessar, S. Koshitani and M. Linckelmann. In their theorem [5, Theorem 1.1], they obtain a better condition than ours since they assume that $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$ which we do not assume.

The following theorem shows that $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable in certain cases.

Theorem 1.4. *Let G be a finite group, P a p-subgroup of G and Q a fully normalized subgroup of P. Suppose that $\mathcal{F}_P(G)$ is saturated. Moreover, we assume that there is a subgroup H_Q of $N_G(Q)$ satisfying the following two conditions:*