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Numerical semigroups II: Pseudo-symmetric AA-semigroups[☆]

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In this work we consider the general numerical AA-semigroup, i.e., semigroups consisting of all non-negative integer linear combinations of relatively prime positive integers of the form $a, a+d, a+2d, \dots, a+kd, c$. We first prove that, in contrast to arbitrary numerical semigroups, there exists an upper bound for the *type* of AA-semigroups that only depends on the number of generators of the semigroup. We then present two characterizations of *pseudo-symmetric* AA-semigroups. The first one leads to a polynomial time algorithm to decide whether an AA-semigroup is pseudo-symmetric. The second one gives a method to construct pseudo-symmetric AA-semigroups and provides explicit families of pseudo-symmetric semigroups with arbitrarily large number of generators.

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1. Introduction

This paper is a continuation of the paper [3]. For a numerical semigroup $\mathcal{S}$, we recall that the *Frobenius number* $g = g(\mathcal{S})$ is the largest integer not in $\mathcal{S}$, and the *genus* $N = N(\mathcal{S})$ is the number of non-negative integers not in $\mathcal{S}$. The semigroup $\mathcal{S}$ is *symmetric* if

$$\mathcal{S} \cup (g - \mathcal{S}) = \mathbb{Z},$$

where $g - \mathcal{S} = \{g - s \mid s \in \mathcal{S}\}$. The semigroup is *pseudo-symmetric* if the Frobenius number g is *even* and

$$\mathcal{S} \cup (g - \mathcal{S}) = \mathbb{Z} \setminus \{g/2\}. \tag{1}$$

It is well known that $\mathcal{S}$ is symmetric if and only if $g = 2N - 1$. Similarly, it is also known that $\mathcal{S}$ is pseudo-symmetric if and only if

$$g = 2N - 2. \tag{2}$$

We include an easy proof of this result in Section 2 (Lemma 2.2).

Set $\Delta(\mathcal{S}) = 2N - 1 - g$. Then $\mathcal{S}$ is symmetric if and only if $\Delta(\mathcal{S}) = 0$, and pseudo-symmetric if and only if $\Delta(\mathcal{S}) = 1$. The numerical semigroup $\mathcal{S}$ is *irreducible* if it is not the intersection of two strictly larger numerical semigroups. It now follows from [1,2,5] that $\mathcal{S}$ is irreducible if and only if $\Delta(\mathcal{S}) \leq 1$.

For a semigroup $\mathcal{S}$ we set $\mathcal{S}' = \{x \notin \mathcal{S} \mid x + s \in \mathcal{S} \text{ for all } s \in \mathcal{S}\}$. The elements of $\mathcal{S}'$ are usually called *pseudo-Frobenius numbers* and the number of elements of $\mathcal{S}'$ is called the *type* of $\mathcal{S}$ and denoted by $\operatorname{type}(\mathcal{S})$. We notice that g is always a pseudo-Frobenius number. Moreover, by [2, Proposition 2] $\mathcal{S}$ is symmetric if and only if $\mathcal{S}' = \{g\}$, or equivalently, if the type of $\mathcal{S}$ is 1. Also, $\mathcal{S}$ is pseudo-symmetric if and only if $\mathcal{S}' = \{g, g/2\}$, which implies that every pseudo-symmetric semigroup has type 2.

The *Apéry set* of $\mathcal{S}$ with respect to a non-zero $m \in \mathcal{S}$ is defined as

$$\operatorname{Ap}(\mathcal{S}; m) = \{s \in \mathcal{S} \mid s - m \notin \mathcal{S}\}.$$

An Apéry set of a semigroup $\mathcal{S}$ is very difficult to determine in general. This set contains many relevant information about the semigroup. As we shall point out, in Section 2, all the above mentioned parameters related to $\mathcal{S}$ can be expressed in terms of the Apéry set of $\mathcal{S}$ with respect to any non-zero $m \in \mathcal{S}$.

In this paper we focus our attention on numerical AA-semigroups consisting of all non-negative integer linear combinations of relatively prime positive integers $a, a+d, a+2d, \ldots, a+kd, c$, where also a, d, k, c are positive integers. In [4], Rødseth presented "semi-explicit" formulas for $\operatorname{Ap}(\mathcal{S}; a)$, $g(\mathcal{S})$ and $N(\mathcal{S})$ when $\mathcal{S}$ is an AA-semigroup.