Journal of Algebra 472 (2017) 1–14

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Maximal group actions on compact oriented surfaces

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ARTICLE INFO

Article history:
Received 21 December 2015
Available online 14 October 2016
Communicated by E.I. Khukhro

Keywords:
Automorphism
Compact Riemann surface
Mapping class group

ABSTRACT

Suppose S is a compact oriented surface of genus $\sigma \geq 2$ and C_p is a group of orientation preserving automorphisms of S of prime order $p \geq 5$. We show that there is always a finite supergroup $G > C_p$ of orientation preserving automorphisms of S except when the genus of S/C_p is minimal (or equivalently, when the number of fixed points of C_p is maximal). Moreover, we exhibit an infinite sequence of genera within which any given action of C_p on S implies C_p is contained in some finite supergroup and demonstrate for genera outside of this sequence the existence of at least one C_p-action for which C_p is not contained in any such finite supergroup (for sufficiently large σ).

1. Introduction

A finite group G is said to act in an *orientation preserving manner* on a compact oriented surface S of genus $\sigma \geq 2$ if there is an injection

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[1] Partially supported by NSF grant DMS-1157105.

http://dx.doi.org/10.1016/j.jalgebra.2016.10.004

$$\epsilon\colon G \hookrightarrow \mathrm{Homeo}^+(S)$$

from G into the group of orientation preserving homeomorphisms. We denote such an action by the ordered pair (G, ϵ), though when unambiguous we write simply G. Two actions (G, ϵ_1), (G, ϵ_2) are said to be *topologically equivalent* if their images $\epsilon_1(G)$ and $\epsilon_2(G)$ are conjugate in $\mathrm{Homeo}^+(S)$.

In the following, we determine when a cyclic group C_p of prime order $p \geq 5$ of orientation preserving homeomorphisms of a surface S is *finitely maximal*, meaning there is no proper finite supergroup $G \leq \mathrm{Homeo}^+(S)$ containing C_p. We show that when such an action exists the genus of S/C_p is minimal (or equivalently, the number of fixed points of the C_p-action is maximal). Following this we show that, for sufficiently large genus, there exists a finitely maximal C_p-action on a surface of genus σ if and only if $\sigma \not\equiv \dfrac{p-3}{2} \left(\mathrm{mod}\, \dfrac{p-1}{2}\right)$.

Though an interesting problem in its own right, there are a number of other motivations for this work. For example, in the context of the moduli space $\mathcal{M}_\sigma$ of compact Riemann surfaces of genus σ, there is widespread interest in describing the branch locus, $\mathcal{B}_\sigma$, which is the subset of $\mathcal{M}_\sigma$ of surfaces with non-trivial automorphisms. We define $\mathcal{M}_\sigma^{(G,\epsilon)} \subset \mathcal{M}_\sigma$ to be the set of surfaces whose full group of conformal automorphisms is topologically equivalent to (G, ϵ), and $\overline{\mathcal{M}}_\sigma^{(G,\epsilon)}$ to be the set of surfaces whose full group of conformal automorphisms contains (G, ϵ). In [5], Broughton showed that the sets $\{\mathcal{M}_\sigma^{(G,\epsilon)}\}$ form a stratification of $\mathcal{B}_\sigma$ known as the *equisymmetric stratification*. A first step in describing this stratification is distinguishing between $\mathcal{M}_\sigma^{(G,\epsilon)}$ and $\overline{\mathcal{M}}_\sigma^{(G,\epsilon)}$; the following results represent a significant step in this direction for $G = C_p$ as well as extending current work ([2]) on identifying the isolated strata of $\mathcal{B}_\sigma$. For further reading on the branch locus of moduli space, see also [1,3,10,11,14].

This work also has implications for the connections between topological group actions and subgroups of the mapping class group. Specifically, if $\mathfrak{M}_\sigma$ denotes the mapping class group in genus σ, then there is a natural one-to-one correspondence between conjugacy classes of finite subgroups of $\mathfrak{M}_\sigma$ and equivalence classes of finite topological group actions on a smooth oriented surface of genus σ. Moreover, if $H < G$ both act on a surface of genus σ, then we have the corresponding containment in $\mathfrak{M}_\sigma$. As such, our results allow one to determine when a given conjugacy class in $\mathfrak{M}_\sigma$ of subgroups isomorphic to C_p is finitely maximal in $\mathfrak{M}_\sigma$. See [7,19] for other recent work in this area.

Perhaps the most important consequence of the following work is also the most direct one: it contributes significantly to the eventual goal of a complete classification of finitely maximal C_p-actions. Specifically, it was shown in [4] that for sufficiently large σ, the number of distinct quotient genera S/C_p for C_p-actions on a surface S of genus σ is linear in σ (though this can also be derived from Theorem 4 below). Theorem 5 therefore implies that when classifying maximal actions one need only consider a single quotient genus, thereby greatly reducing the complexity of the problem.