Journal of Algebra 472 (2017) 95–114

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Equivariant cohomology and the Varchenko–Gelfand filtration

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ARTICLE INFO

Article history:
Received 31 December 2013
Available online 20 October 2016
Communicated by Gus I. Lehrer

Keywords:
Hyperplane arrangements
Equivariant cohomology
Representation theory
Configuration space
Oriented matroids

ABSTRACT

The cohomology of the configuration space of n points in $\mathbb{R}^3$ is isomorphic to the regular representation of the symmetric group, which acts by permuting the points. We give a new proof of this fact by showing that the cohomology ring is canonically isomorphic to the associated graded of the Varchenko–Gelfand filtration on the cohomology of the configuration space of n points in $\mathbb{R}^1$. Along the way, we give a presentation of the equivariant cohomology ring of the $\mathbb{R}^3$ configuration space with respect to a circle acting on $\mathbb{R}^3$ via rotation around a fixed line. We extend our results to the settings of arbitrary real hyperplane arrangements (the aforementioned theorems correspond to the braid arrangement) as well as oriented matroids.

1. Introduction

Definition 1.1. Let V be a finite dimensional real vector space, and let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in V given by $H_i = \omega_i^{-1}(0)$ for some non-constant affine

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http://dx.doi.org/10.1016/j.jalgebra.2016.10.010
0021-8693/

linear form $\omega_i : V \to \mathbb{R}$. Let $\overline{\omega_i}$ be the associated linear map. Define affine linear maps $\omega_{i,k} : V^k \to \mathbb{R}^k$ by

$$\omega_{i,k}(v_1, \dots, v_k) = (\omega_i(v_1), \overline{\omega_i}(v_2), \dots, \overline{\omega_i}(v_k)).$$

The space $M_k(\mathcal{A})$ is defined to be the complement of the union of the affine subspaces

$$\omega_{i,k}^{-1}(0, 0, \dots, 0).$$

When $k = 1$, this is just the complement of the arrangement. When $k = 2$, this is isomorphic to the complement of the complexified arrangement. When $\mathcal{A}$ is the braid arrangement $\mathcal{B}_n$, $M_k(\mathcal{A})$ is the configuration space of n ordered points in $\mathbb{R}^k$.

Consider the ring[1] $H^0(M_1(\mathcal{A}))$ of locally constant functions on $M_1(\mathcal{A})$. Varchenko and Gelfand defined a filtration of this ring via Heaviside functions. Let

$$H_i^+ = \{v \in V \mid \omega_i(v) > 0\},$$

and let

$$H_i^- = \{v \in V \mid \omega_i(v) < 0\}.$$

Define the Heaviside function $x_i \in H^0(M_1(\mathcal{A}))$ by putting

$$x_i(v) = \begin{cases} 1 & v \in H_i^+ \\ 0 & v \in H_i^-. \end{cases}$$

Proposition 1.2. *[1, Thm 4.5] Consider the map $\psi : \mathbb{Q}[e_1, \dots, e_n] \to H^0(M_1(\mathcal{A}))$ taking e_i to x_i, and let $\mathcal{I}_1$ be the kernel. This map is surjective, and $\mathcal{I}_1$ is generated by the following relations:*

(1) $e_i^2 - e_i$

(2) $\prod_{i \in S^+} e_i \prod_{j \in S^-} (e_j - 1)$ *if* $\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing$

(3) $\prod_{i \in S^+} e_i \prod_{j \in S^-} (e_j - 1) - \prod_{i \in S^+} (e_i - 1) \prod_{j \in S^-} e_j$

if $\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing$ *and* $\bigcap_{i \in (S^+ \cup S^-)} H_i \neq \varnothing$.

Remark 1.3. Note that only families (1) and (2) are necessary to generate $\mathcal{I}_1$. However, in the case that $\mathcal{A}$ is central, only families (1) and (3) are necessary to generate $\mathcal{I}_1$

[1] All cohomology rings in this paper will be taken with coefficients in $\mathbb{Q}$.