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Small intersections of principal blocks ☆

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ABSTRACT

In this note, it is shown that a finite group G is solvable if for each odd prime divisor p of $|G|$, $|\mathrm{Irr}(B_0(G)_2) \cap \mathrm{Irr}(B_0(G)_p)| \leq 2$, where $\mathrm{Irr}(B_0(G)_p)$ is the set of complex irreducible characters of the principal p-block $B_0(G)_p$ of G. Also, the structure of such groups is investigated. Examples show that the bound 2 is best possible.

1. Introduction

The block distributions of complex irreducible characters of a finite group have been recently investigated across distinct primes in the modular representation theory of finite groups. For instance, G. Navarro and W. Willems [25] considered the question when a p-block is a q-block. After a few years, C. Bessenrodt, G. Malle and J. B. Olsson [2] intro-

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duced the concept of separability of characters by blocks, and G. Navarro, A. Turull and T. R. Wolf [24] discussed solvable groups that are block separated. Later, C. Bessenrodt and the second author [3] investigated separations of characters by principal blocks of a finite group. It turns out that the intersections of principal blocks of a finite group have an influence on the structure of the group.

Let G be a finite group and $\mathrm{Irr}(G)$ the set of complex irreducible characters of G. Let p and q be two distinct prime divisors of $|G|$. Denote by $B_0(G)_p$ and $\mathrm{Irr}(B_0(G)_p)$ the principal p-block of G and the set of complex irreducible characters of G contained in $B_0(G)_p$, respectively. In this note, we focus on finite groups G with small principal block intersections and provide a criterion for a finite group to be solvable.

Theorem 1.1. *Let G be a finite group. Suppose that for each odd prime divisor p of $|G|$, $|\mathrm{Irr}(B_0(G)_2) \cap \mathrm{Irr}(B_0(G)_p)| \leq 2$. Then G is solvable.*

With an application of Theorem 1.1, we have the following result.

Theorem 1.2. *Let G be a finite group. Suppose that for each odd prime divisor p of $|G|$, $|\mathrm{Irr}(B_0(G)_2) \cap \mathrm{Irr}(B_0(G)_p)| \leq 2$. Then G is 2-nilpotent.*

Proof. By Theorem 1.1, G is solvable. If $|\mathrm{Irr}(B_0(G)_2) \cap \mathrm{Irr}(B_0(G)_p)| = 1$ then by [3, Proposition 2.1], $G = O_{2'}(G)O_{p'}(G)$; and if $|\mathrm{Irr}(B_0(G)_2) \cap \mathrm{Irr}(B_0(G)_p)| = 2$ then since $G/O_{p'}(G)$ has a unique p-block, we have

$$|\mathrm{Irr}(G/O_{2'}(G)O_{p'}(G))| = |\mathrm{Irr}(G/O_{2'}(G)) \cap \mathrm{Irr}(G/O_{p'}(G))| \tag{1.1}$$

$$= |\mathrm{Irr}(B_0(G)_2) \cap \mathrm{Irr}(B_0(G)_p)| \tag{1.2}$$

$$= 2 \tag{1.3}$$

so that $G/O_{2'}(G)O_{p'}(G)$ has exactly two conjugacy classes. Note that C_2 is the only group with two conjugacy classes up to isomorphism. This is because if a finite group H has exactly two conjugacy classes then $H - \{1\}$ is a conjugacy class, which implies $(|H|-1) \mid |H|$ and so $|H| = 2$. Therefore, $|G/O_{2'}(G)O_{p'}(G)| = 2$. By the choice of p, we conclude that $O_{2'}(G)$ contains all Sylow p-subgroups of G for each odd prime divisor p of $|G|$. Hence $O_{2'}(G)$ is a normal 2-complement of G, and thus G is 2-nilpotent. $\square$

In the following we consider an important special case of Theorem 1.1. For the purpose of convenience, we define

$$\alpha_{p,q}(G) = |\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q)|$$

and

$$\alpha(G) = \max\{\alpha_{p,q}(G) \mid p, q \text{ are distinct prime divisors of } |G|\}.$$