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On the shape of possible counterexamples to the Jacobian Conjecture

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A B S T R A C T

We improve the algebraic methods of Abhyankar for the Jacobian Conjecture in dimension two and describe the shape of possible counterexamples. We give an elementary proof of the result of Heitmann in [5], which states that $\gcd(\deg(P), \deg(Q)) \geq 16$ for any counterexample (P, Q). We also prove that $\gcd(\deg(P), \deg(Q)) \neq 2p$ for any prime p.

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Introduction

Let K be a field of characteristic zero. The Jacobian Conjecture (JC) in dimension two, stated by Keller in [7], says that any pair of polynomials $P, Q \in L := K[x,y]$ with $[P,Q] := \partial_x P \partial_y Q - \partial_x Q \partial_y P \in K^{\times}$ defines an automorphism of $K[x,y]$.

In this paper we improve the algebraic methods of Abhyankar describing the shape of the support of possible counterexamples. We use elementary algebraic methods combined with basic discrete analytic geometry on the plane, i.e. on the points $\mathbb{N}_0 \times \mathbb{N}_0$ in the case of $L = K[x,y]$ and in $\frac{1}{l}\mathbb{Z} \times \mathbb{N}_0$ in the case of $L^{(l)} := K[x^{\pm\frac{1}{l}}, y]$.

The first innovation is a definition of the directions and an order relation on them, based on the crossed product of vectors, which simplifies substantively the treatment of consecutive directions associated with the Newton polygon of Jacobian pairs. It is related to [5, Lemma 1.15] and enables us to simplify substantially the treatment of the Newton polygon and its edges (compare with [2, 7.4.14]).

The second innovation lies in the use of the polynomial F with $[F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P)$, obtained in Theorem 2.6 for a given Jacobian pair (P,Q). This element can be traced back to 1975 in [6]. There also appears the element $G_0 \in K[P,Q]$, which becomes important in the proof of our Proposition 7.1. The polynomial F mentioned above is well known and used by many authors, see for example [6,10] and [11, 10.2.8] (together with [11, 10.2.17 i)]). In Theorem 2.6, we add some geometric statements on the shape of the supports, especially about the endpoints (called st and en) associated to an edge of the Newton Polygon. In [5, Proposition 1.3] some of these statements, presented in an algebraic form, can be found.

We will apply different endomorphisms in order to deform the support of a Jacobian pair. Opposed to most of the authors working in this area [5,12,9], we remain all the time in L (or $L^{(l)}$). In order to do this we use the following very simple expression of the change of the Jacobian under an endomorphism $\varphi\colon L \to L$ (or $L \to L^{(l)}$, or $L^{(l)} \to L^{(l)}$):

$$[\varphi(P), \varphi(Q)] = \varphi([P,Q])[\varphi(x), \varphi(y)].$$

Another key ingredient is the concept of regular corners and its classification, which we present in Section 5. The geometric fact that certain edges can be cut above the diagonal, Proposition 5.16, was already known to Joseph and used in [6, Theorem 4.2], in order to prove the polarization theorem.

In Section 6 we give an elementary proof of a result of [5]: If

$$B := \begin{cases} \infty & \text{if the jacobian conjecture is true,} \\ \min\bigl(\gcd(v_{1,1}(P), v_{1,1}(Q))\bigr) & \text{if it is false, where } (P,Q) \text{ runs} \\ & \text{on the counterexamples,} \end{cases}$$

then $B \geq 16$. In spite of Heitmann's assertion “Nothing like this appears in the literature but results of this type are known by Abhyankar and Moh and are easily inferred from