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The relative modular object and Frobenius extensions of finite Hopf algebras

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ABSTRACT

For a certain kind of tensor functor $F : \mathcal{C} \to \mathcal{D}$, we define the relative modular object $\chi_F \in \mathcal{D}$ as the "difference" between a left adjoint and a right adjoint of F. Our main result claims that, if $\mathcal{C}$ and $\mathcal{D}$ are finite tensor categories, then χ_F can be written in terms of a categorical analogue of the modular function on a Hopf algebra. Applying this result to the restriction functor associated to an extension A/B of finite-dimensional Hopf algebras, we recover the result of Fischman, Montgomery and Schneider on the Frobenius type property of A/B. We also apply our results to obtain a "braided" version and a "bosonization" version of the result of Fischman et al.

1. Introduction

Frobenius-type properties of extensions of Hopf algebras and of related algebras have been studied extensively; see, *e.g.*, [27,7,15,16,19,20]. Fischman, Montgomery and Schneider [15] showed that the Frobenius property of an extension of finite-dimensional Hopf algebras is controlled by their modular functions. The aim of this paper is to formulate

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and prove a generalization of their result in the setting of finite tensor categories, a class of tensor categories including the representation category of a finite-dimensional Hopf algebra.

To explain our results in more detail, we briefly recall the above-mentioned result of Fischman et al. Recall that, for a finite-dimensional Hopf algebra H over a field k, the (right) *modular function* $\alpha_H : H \to k$ (also called the distinguished grouplike element in literature) is defined by

$$h \cdot \Lambda = \alpha_H(h)\Lambda \quad (h \in H), \tag{1.1}$$

where $\Lambda \in H$ is a non-zero right integral. For an extension A/B of finite-dimensional Hopf algebras over k (meaning that A is such a Hopf algebra and B is a Hopf subalgebra of A), the *relative modular function* $\chi = \chi_{A/B}$ and the *relative Nakayama automorphism* $\beta = \beta_{A/B}$ of A/B are defined respectively by

$$\chi(b) = \alpha_A(b_{(1)})\alpha_B(S(b_{(2)})) \quad \text{and} \quad \beta(b) = \chi(b_{(1)})b_{(2)} \tag{1.2}$$

for $b \in B$, where S is the antipode of B and $\Delta(b) = b_{(1)} \otimes b_{(2)}$ is the comultiplication of b in the Sweedler notation [15, Definition 1.6]. They showed that A/B is a *β-Frobenius extension*, *i.e.*, A is finitely-generated and projective as a right B-module (by the Nichols–Zoeller theorem) and there is an isomorphism

$${}_B A_A \cong {}_\beta \mathrm{Hom}_B(A_B, B_B) \tag{1.3}$$

of B-A-bimodules [15, Theorem 1.7], where ${}_\beta(-)$ means the left B-module obtained by twisting the action of B by β.

To formulate this result in a category-theoretical setting, we consider the restriction functor $\mathrm{res}^A_B : \text{mod-}A \to \text{mod-}B$ between the categories of right modules. By the standard argument and the Nichols–Zoeller theorem, the functors

$$L := (-) \otimes_B A \quad \text{and} \quad R := \mathrm{Hom}_B(A, -) \cong (-) \otimes_B \mathrm{Hom}_B(A, B) \tag{1.4}$$

are a left adjoint and a right adjoint of res^A_B, respectively. Hence (1.3) says that the relative modular function measures the "difference" between L and R. Here we remark that res^A_B is in fact a tensor functor. Thus, we are led to the problem of studying the "difference" between a left adjoint and a right adjoint of a tensor functor.

Based on this observation, we consider a tensor functor $F : \mathcal{C} \to \mathcal{D}$ having a left adjoint L and a right adjoint R. It turns out that, under certain assumptions, there exists a unique (up to isomorphism) object $\chi_F \in \mathcal{D}$ such that $L \cong R(\chi_F \otimes -)$. Our main result is that if $\mathcal{C}$ and $\mathcal{D}$ are finite tensor categories, then the object χ_F is expressed in terms of the category-theoretical analogue of the modular function introduced by Etingof, Nikshych and Ostrik [13]. Applying this result to res^A_B, we recover the above-mentioned result of Fischman et al. We also apply our results to obtain some variants of their result.