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Description of Galois unipotent extensions ☆

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A B S T R A C T

Given an arbitrary field F, we describe all Galois extensions L/F whose Galois groups are isomorphic to the group of upper triangular unipotent 4-by-4 matrices with entries in the field of two elements.

1. Introduction

Let G be a finite group, and let F be an arbitrary field. A fundamental problem in Galois theory is to describe all Galois extensions L/F whose Galois groups are isomorphic to group G. It is desirable to describe such families of extensions using invariants of L/F

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which depend only on the base field F. If G is abelian then this is possible by the theories of Kummer and Artin–Schreier's extension, and classical work of A. Albert and D.J. Saltman. Moreover this description is elegant, simple and useful. It is known that there are some other very interesting and useful explicit constructions of Galois extensions L/F with prescribed Galois group G. See for example, [6], [7, Chapters 5–6], [12, Chapters 2, 5–7], [14,15,22,25]. However the simplicity and generality of the descriptions of Kummer and Artin–Schreier's extension seem to be unmatched.

Recall that for each natural number n, $\mathbb{U}_n(\mathbb{F}_p)$ is the group of upper triangular $n \times n$-matrices with entries in $\mathbb{F}_p$ and diagonal entries 1. In a recent development of Massey products in Galois cohomology, it was recognized that Galois extensions L/F with $\mathrm{Gal}(L/F) \simeq \mathbb{U}_n(\mathbb{F}_p)$ play a very special role in Galois theory of p-extensions. (See [3,4,9,2,8,17,18,20,21].) Moreover the works above reveal some surprising depth and simplicity of analysis of these extensions. The main purpose of our paper is to describe all Galois extensions L/F with $\mathrm{Gal}(L/F) \simeq \mathbb{U}_4(\mathbb{F}_2)$ over any given field F. Our main results are Theorem 2.8 and Theorem 4.7. We also show that a similar description is valid for Galois extensions with Galois group isomorphic to $\mathbb{U}_3(\mathbb{F}_2)$ over an arbitrary field. (Note that $\mathbb{U}_3(\mathbb{F}_2)$ is isomorphic to the dihedral group of order 8.)

Beside of their intrinsic value, these simple descriptions of Galois extensions L/F with $\mathrm{Gal}(L/F) \simeq \mathbb{U}_4(\mathbb{F}_2)$ are expected to play a significant role in an induction approach to the construction of Galois extensions L/F with $\mathrm{Gal}(L/F) \simeq \mathbb{U}_n(\mathbb{F}_2)$ for $n \geq 2$, and for a possible proof of the Vanishing n-Massey Conjecture for absolute Galois groups of fields. (See [17,21].) Also this description should be useful for establishing the Kernel n-Unipotent Conjecture for absolute Galois groups of fields and $p = 2$. This would be a very interesting extension of the work of [16,29]. (See also [5,18].)

Indeed a natural program for solving the Vanishing n-Massey Conjecture for absolute Galois groups of fields uses Theorem 2.4 in [2]. This theorem reduces the problem to solving a certain Galois embedding problem induced from the central extension

$$1 \to \mathbb{F}_p \to \mathbb{U}_{n+1}(\mathbb{F}_p) \to \bar{\mathbb{U}}_{n+1}(\mathbb{F}_p) \to 1,$$

where $\bar{\mathbb{U}}_{n+1}(\mathbb{F}_p)$ is the quotient of $\mathbb{U}_{n+1}(\mathbb{F}_p)$ by its center. The most interesting and difficult task is in fact to find a construction of Galois extensions L/F with $\mathrm{Gal}(L/F) \simeq \mathbb{U}_{n+1}(\mathbb{F}_p)$ that solve the embedding problem. Because $\mathbb{U}_{n+1}(\mathbb{F}_p)$ contains copies of $\mathbb{U}_n(\mathbb{F}_p)$, one can consider to use induction on n. This program was realized in [21] in the case $n = 3$. Here the knowledge of the explicit construction of Galois extensions with Galois group $\mathbb{U}_3(\mathbb{F}_p)$ was crucial. A successful implementation of this program also in the case $n = 4$ may reveal the induction procedure which is valid in general. Therefore the knowledge of Galois extensions L/F with $\mathrm{Gal}(L/F) \simeq \mathbb{U}_4(\mathbb{F}_2)$ seems to be important for the implementation of this program.

Further possible applications of this work can be related to an extension of the study of Redei symbols and also the study of 2-Hilbert towers. (See [1,13].)