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Ascending chains of finitely generated subgroups

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ABSTRACT

We show that a nonempty family of n-generated subgroups of a pro-p group has a maximal element. This suggests that 'Noetherian Induction' can be used to discover new features of finitely generated subgroups of pro-p groups. To demonstrate this, we show that in various pro-p groups Γ (e.g. free pro-p groups, nonsolvable Demushkin groups) the commensurator of a finitely generated subgroup $H \neq 1$ is the greatest subgroup of Γ containing H as an open subgroup. We also show that an ascending chain of n-generated subgroups of a limit group must terminate (this extends the analogous result for free groups proved by Takahasi, Higman, and Kapovich–Myasnikov).

1. Introduction

Chain conditions play a prominent role in Algebra. A good example is the variety of results on Noetherian rings and their modules. In this work we consider chain conditions on profinite groups. All the group-theoretic notions considered for these groups should be understood in the topological sense, i.e. subgroups are closed, homomorphisms are continuous, generators are topological, etc. Fix once and for all a prime number p. The

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ascending chain condition on finitely generated subgroups does not hold for pro-p groups in general, and our first result is some kind of remedy for this.

Proposition 1.1. *Let $n \in \mathbb{N}$, let Γ be a pro-p group, and let $\mathcal{F} \neq \emptyset$ be a family of n-generated subgroups of Γ. Then $\mathcal{F}$ has a maximal element with respect to inclusion.*

As illustrated in the sequel, this simple result unveils new properties of pro-p groups and their finitely generated subgroups. An example is the following theorem, for which we need some definitions. We say that a pro-p group Γ has a ***Hereditarily Linearly Increasing Rank*** (the word 'rank' is to be understood in the sense of a minimal number of generators) if for every finitely generated subgroup $H \leq_c \Gamma$ there exists an $\epsilon > 0$ such that for any open subgroup $U \leq_o H$ we have

$$d(U) \geq \max\{d(H), \epsilon(d(H) - 1)[H : U]\} \tag{1.1}$$

where $d(K)$ stands for the smallest cardinality of a generating set for the pro-p group K. That is, our definition says that the minimal number of generators of finite index subgroups of H grows monotonically, and linearly (unless H is procyclic) as a function of the index. Examples of groups with this property include free pro-p groups, nonsolvable Demushkin groups, and groups from the class $\mathcal{L}$ all of whose abelian subgroups are procyclic (see [30]).

The linear growth of the number of generators of subgroups of H as a function of their index means that the rank gradient of H, defined by

$$\inf_{U \leq_o H} \frac{d(U) - 1}{[H : U]} \tag{1.2}$$

is positive. The rank gradient is at the focus of much recent research in both profinite and abstract group theory, as can be seen, for instance, from [1,2,7,15,19,22,24,28,29].

Let us introduce some more definitions and notation. Subgroups H_1, H_2 of a profinite group Γ are said to be commensurable if $H_1 \cap H_2$ is open in both H_1 and H_2. Given a subgroup $H \leq_c \Gamma$, the commensurator of H in Γ, that is, the set of $\gamma \in \Gamma$ for which H and $\gamma H \gamma^{-1}$ are commensurable, is denoted by $\mathrm{Comm}_\Gamma(H)$. The commensurator is an abstract subgroup of Γ. We define the family of 'finite extensions' of H in Γ by $\mathcal{F} := \{R \leq_c \Gamma \mid H \leq_o R\}$. Following [26], we say that $M \in \mathcal{F}$ is the root of H (and write $M = \sqrt{H}$) if M is the greatest element in $\mathcal{F}$ with respect to inclusion. Note that $\mathcal{F}$ may fail to have a greatest element, so H does not necessarily have a root.

Theorem 1.2. *Let Γ be a pro-p group with a hereditarily linearly increasing rank, and let $1 \neq H \leq_c \Gamma$ be a finitely generated subgroup. Then $\mathrm{Comm}_\Gamma(H) = \sqrt{H}$, and the action of any $\sqrt{H} \lneq_c L \leq_c \Gamma$ by multiplication from the left on L/H is faithful.*

In particular, there are only finitely many subgroups of Γ that contain H as an open subgroup (and $\mathrm{Comm}_\Gamma(H)$ is one of these). Thus, H is also an open subgroup of its