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Motzkin monoids and partial Brauer monoids

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A B S T R A C T

We study the partial Brauer monoid and its planar submonoid, the Motzkin monoid. We conduct a thorough investigation of the structure of both monoids, providing information on normal forms, Green's relations, regularity, ideals, idempotent generation, minimal (idempotent) generating sets, and so on. We obtain necessary and sufficient conditions under which the ideals of these monoids are idempotent-generated. We find formulae for the rank (smallest size of a generating set) of each ideal, and for the idempotent rank (smallest size of an idempotent generating set) of the idempotent-generated subsemigroup of each ideal; in particular, when an ideal is idempotent-generated, the rank and idempotent rank are equal. Along the way, we obtain a number of results of independent interest, and we demonstrate the utility of the semigroup theoretic approach by applying our results to obtain new proofs of some important representation theoretic results concerning the corresponding diagram algebras, the partial (or rook) Brauer algebra and Motzkin algebra.

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1. Introduction

There are many reasons to study the monoids of the title. On the one hand, *diagram algebras* (including Brauer algebras [12], partial Brauer algebras [43,60], Temperley–Lieb algebras [72], Motzkin algebras [9], rook monoid algebras [44], partition algebras [54,59], etc.) are ubiquitous in representation theory and statistical mechanics, and several recent approaches to diagram algebras via *diagram monoids* and *twisted semigroup algebras* have proved extremely successful [11,16,17,21,22,45,57,74]. On the other hand, diagram monoids are of direct interest to semigroup theorists for a range of reasons. For one thing, diagram monoids are closely related to several important transformation semigroups; indeed, the partition monoids contain isomorphic copies of the full transformation semigroup and the symmetric inverse monoid [21,24,28]. For another, diagram monoids provide natural examples of *regular $*$-semigroups* [69], a variety of semigroups contained strictly between the varieties of regular and inverse semigroups (see Section 2 for definitions); thanks to the $*$-regular structure of the partition monoids, it is now known that any semigroup embeds in an idempotent-generated regular $*$-semigroup [24], and that any finite semigroup embeds in a 2-generator regular $*$-semigroup [23]. Diagram monoids have also played a large role in the development of the theory of pseudovarieties of finite semigroups [4–7]. Finally, and of particular significance to the current work, algebraic studies of diagram monoids have led to incredibly rich combinatorial structures [16,17,25].

The current article continues in the combinatorial theme of a previous paper of the second and third named authors [25], which, in turn, took its inspiration from a number of foundational papers of John Howie on combinatorial aspects of finite full transformation semigroups. In [48], Howie showed that the singular ideal of a finite full transformation semigroup is generated by its idempotents. In subsequent work, Howie and various collaborators calculated the *rank* (minimal size of a generating set) and *idempotent rank* (minimal size of an idempotent generating set) of this singular ideal, classified the minimal (in size) idempotent generating sets, and extended these results to arbitrary ideals [35,49,51,52]; a prominent role was played by certain well-known number sequences, such as binomial coefficients and Stirling numbers. Analogous results have been obtained for many other important families of semigroups, such as full linear monoids [26,29,38], endomorphism monoids of various algebraic structures [15,18,30,31,37], sandwich semigroups [13,14] and, more recently, certain families of diagram monoids [22,24,25,58].

The current article mainly concerns the *partial Brauer monoid* $\mathcal{PB}_n$ and the *Motzkin monoid* $\mathcal{M}_n$ (see Section 2 for definitions). There is a growing body of literature on partial Brauer monoids and algebras[1]; see for example [7,16,43,46,55,60,64–66]. Motzkin monoids and algebras are a more recent phenomenon [7,9,17,46,66,70], and are *planar*

[1] Partial Brauer monoids and algebras are also known in the literature as *rook Brauer* monoids and algebras.