Journal of Algebra 471 (2017) 319–347

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On weak commutativity in groups

Dessislava Kochloukova [a,b,*], Said Sidki [a,b]

[a] State University of Campinas (UNICAMP), Campinas, Brazil
[b] University of Brasília (UnB), Brasília, Brazil

A R T I C L E   I N F O

Article history:
Received 2 March 2016
Available online 30 August 2016
Communicated by E.I. Khukhro

MSC:
primary 20J05
secondary 20F05

Keywords:
Group theory
Finite presentability
Weak commutativity
Homological type FPn

A B S T R A C T

For a group G we study homological and homotopical properties of the group $\chi(G) = \langle G, G^\psi \mid [g, g^\psi] = 1 \text{ for } g \in G\rangle$. In particular, we show that the operator χ preserves the soluble of type FP_∞ property.

1. Introduction

In [30] Sidki associated to an arbitrary group G a new group $\chi(G)$ which is defined by two isomorphic copies of G and satisfies some natural commutator relations. It turned out that for G finite, the group $\chi(G)$ is always finite and for an arbitrary G, surprisingly $\chi(G)$ has a subquotient that is isomorphic to the Schur multiplier $H_2(G, \mathbb{Z})$. By definition

$$\chi(G) = \langle G, G^\psi \mid [g, g^\psi] = 1 \text{ for } g \in G\rangle,$$

* Corresponding author.
E-mail addresses: desi@ime.unicamp.br (D. Kochloukova), s.n.sidki@mat.unb.br (S. Sidki).

http://dx.doi.org/10.1016/j.jalgebra.2016.08.020
0021-8693/

where $\psi : G \to G^{\psi}$ is an isomorphism of groups. As shown in [30, Thm. C] if $\mathcal{P}$ is one of the following classes of groups : finite π-groups, where π is a set of primes; finite nilpotent groups; solvable groups; perfect groups and $G \in \mathcal{P}$ then $\chi(G) \in \mathcal{P}$. Later on Gupta, Rocco and Sidki showed in [19] that if G is finitely generated nilpotent then $\chi(G)$ is nilpotent and found bounds on the nilpotency class of $\chi(G)$. Recently Lima and Oliveira proved that if G is polycyclic-by-finite then $\chi(G)$ is polycyclic-by-finite [22]. Our first result on $\chi(G)$ considers the case when G is soluble of homological type FP_∞. Recall that a group G is of type FP_m if there is a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ with finitely generated projectives in dimension $\leq m$ and G is of type FP_∞ if it is of type FP_m for every m. Our first theorem shows that the operator χ preserves the soluble of type FP_∞ property.

Theorem A. *Let G be a soluble group of type FP_∞. Then $\chi(G)$ is a soluble group of type FP_∞.*

As shown by Sidki in [30] the group $\chi(G)$ has two normal abelian subgroups $R(G)$ and $W(G)$ such that $\chi(G)/W(G)$ is isomorphic to a subgroup of $G \times G \times G$ that contains the commutator subgroup and $W(G)/R(G) \simeq H_2(G, \mathbb{Z})$. The crucial point of the proof of Lima and Oliveira in [22] is that if G is polycyclic-by-finite then $W(G)$ is finitely generated. We generalize the main idea of this proof by introducing a more complicated homological argument that uses spectral sequences, see Theorem 6.1, which is a homological version of Martinez-Perez's result [23, Thm. C]. This enables us to prove the following theorem.

Theorem B. *Let G be a group of type FP_2 such that G'/G'' is finitely generated. Then $W(G)$ is finitely generated.*

As a corollary we obtain

Corollary C. *If G is finitely presented and G'/G'' is finitely generated then $\chi(G)$ is finitely presented.*

Recall that a group G has homotopical type F_m if there is a $K(G, 1)$ complex with finite m-skeleton. In particular a group is of type F_2 if and only if it is finitely presented. For $m \geq 2$ a group is of type F_m if and only if it is FP_m and finitely presented. Using the link between Σ-theory and results of Bieri and Renz [11,29] on homological/homotopical properties of subgroups containing the commutator subgroup, we establish the following result.

Theorem D. *Let G be a group of type F_k (respectively type FP_k) and the commutator subgroup G' has type F_s (respectively type FP_s). Suppose that $k \leq 3s+2$. Then $\chi(G)/W(G)$ has type F_k (respectively type FP_k). Furthermore if G'/G'' is finitely generated and $k \geq 2$ we have that $\chi(G)$ has type F_k (respectively type FP_k).*