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Tilting bundles on toric Fano fourfolds

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ABSTRACT

This paper constructs tilting bundles obtained from full strong exceptional collections of line bundles on all smooth 4-dimensional toric Fano varieties. The tilting bundles lead to a large class of explicit Calabi–Yau-5 algebras, obtained as the corresponding rolled-up helix algebra. A database of the full strong exceptional collections can be found in the package *QuiversToricVarieties* for the computer algebra system *Macaulay2*.

1. Introduction

Let X be a smooth variety over $\mathbb{C}$ and let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on X. A tilting object $\mathcal{T} \in \mathcal{D}^b(X)$ is an object such that $\mathrm{Hom}^i(\mathcal{T}, \mathcal{T}) = 0$ for $i \neq 0$ and $\mathcal{T}$ generates $\mathcal{D}^b(X)$. If such a $\mathcal{T}$ exists, then tilting theory provides an equivalence of triangulated categories between $\mathcal{D}^b(X)$ and the bounded derived category $\mathcal{D}^b(A)$ of finitely generated right modules over the algebra $A = \mathrm{End}(\mathcal{T})$ via the adjoint functors

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$$\begin{tikzcd} \mathcal{D}^b(X) \arrow[rr, bend left, "\mathbf{R}\operatorname{Hom}_X(\mathcal{T},-)"] & & \mathcal{D}^b(A) \arrow[ll, bend left, "(-)\overset{\mathbb{L}}{\otimes}_A \mathcal{T}"] \end{tikzcd}$$

If X is also projective then one can use a full strong exceptional collection to obtain a tilting object; a full strong exceptional collection of sheaves $\{E_i\}_{i\in I}$ defines a tilting sheaf $\mathcal{T} := \bigoplus_{i\in I} E_i$ and conversely, the non-isomorphic summands in a tilting sheaf determine a full strong exceptional collection. The classical example of a tilting sheaf was provided by Beĭlinson [6], who showed that $\mathcal{O} \oplus \mathcal{O}(1) \oplus \ldots \oplus \mathcal{O}(n)$ is a tilting bundle for $\mathbb{P}^n$.

The combinatorial nature of toric varieties makes it feasible to check whether a collection of line bundles on a smooth projective toric variety is full strong exceptional, in which case one can construct the resulting endomorphism algebra explicitly. Smooth toric Fano varieties are of particular interest; there are a finite number of these varieties in each dimension and they have been classified in dimension 3 by Watanabe–Watanabe and Batyrev [41,3], dimension 4 by Batyrev and Sato [4,37], dimension 5 by Kreuzer–Nill [29], whilst Øbro [31] provided a general classification algorithm. King [28] has exhibited full strong exceptional collections of line bundles for the 5 smooth toric Fano surfaces, and by building on work by Bondal [8], Costa–Miró-Roig [13] and Bernardi–Tirabassi [11], Uehara [39] provided full strong exceptional collections of line bundles for the 18 smooth toric Fano threefolds. The main theorem of this paper is as follows:

Theorem 7.4. *Let X be one of the* 124 *smooth toric Fano fourfolds. Then one can construct explicitly a full strong exceptional collection of line bundles on X, a database of which is contained in the computer package* QuiversToricVarieties *[35] for* Macaulay2 *[22].*

In addition to low-dimensional smooth toric Fano varieties, other classes of toric varieties have been shown to have full strong exceptional collections of line bundles – for example, see [13,18,30]. Kawamata [26] showed that every smooth toric Deligne–Mumford stack has a full exceptional collection of sheaves, but we note that these collections are not shown to be strong, nor do they consist of bundles. It is important to note that the existence of full strong exceptional collections of line bundles is rare; Hille–Perling [24] constructed smooth toric surfaces that do not have such collections. Even when only considering smooth toric Fano varieties, there exist examples in dimensions ≥ 419 that do not have full strong exceptional collections of line bundles, as demonstrated by Efimov [19].

The tilting bundle we construct on each smooth toric Fano variety determines a tilting bundle on the total space of the canonical bundle ω_X:

Theorem 7.8. *Let X be an n-dimensional smooth toric Fano variety for $n \leq 4$, $\mathcal{L} = \{L_0, \ldots, L_r\}$ be the full strong exceptional collection on X from the database and $\pi\colon Y :=$*