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How many hypersurfaces does it take to cut out a Segre class?

Paolo Aluffi

Mathematics Department, Florida State University, Tallahassee, FL 32306, USA

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ABSTRACT

We prove an identity of Segre classes for zero-schemes of compatible sections of two vector bundles. Applications include bounds on the number of equations needed to cut out a scheme with the same Segre class as a given subscheme of (for example) a projective variety, and a 'Segre–Bertini' theorem controlling the behavior of Segre classes of singularity subschemes of hypersurfaces under general hyperplane sections. These results interpolate between an observation of Samuel concerning multiplicities along components of a subscheme and facts concerning the integral closure of corresponding ideals. The Segre–Bertini theorem has applications to characteristic classes of singular varieties. The main results are motivated by the problem of computing Segre classes explicitly and applications of Segre classes to enumerative geometry.

1. Introduction

A result from P. Samuel's thesis states that, under mild hypotheses, in computing the multiplicity of a variety Y along a subscheme Z at an irreducible component V of Z we may replace the ideal determined by Z in the local ring $\mathscr{O}_{V,Y}$ by an ideal generated by

E-mail address: aluffi@math.fsu.edu.

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$\operatorname{codim}_V Y$ elements (cf. [16, Theorem 22]). In Fulton–MacPherson intersection theory, the same multiplicity may be defined by means of *Segre classes* [6, §4.3]; it is then natural to ask whether the number of equations needed to define a Segre class may be similarly bounded. This is one of the questions we answer in this note. We work over an algebraically closed field, and our schemes are embeddable in nonsingular varieties. We denote by Z_{red} the reduced scheme supported on Z, and by $s(Z, Y)$ the Segre class of Z in Y.

Theorem 1.1. *Let Y be a pure-dimensional scheme, and let $Z \subseteq Y$ be a closed subscheme. Let X_i, $i = 1, \dots$ be general elements of a linear system cutting out Z.*

(*a*) *Let $Z' := X_1 \cap \cdots \cap X_{\dim Y+1}$. Then ${Z'}_{red} = Z_{red}$, and $s(Z', Y) = s(Z, Y)$.*
(*b*) *Let $Z'' := X_1 \cap \cdots \cap X_{\dim Y}$. Then there exists an open neighborhood Y° of Z in Y such that $(Z'' \cap Y^\circ)_{red} = Z_{red}$, and $s(Z'' \cap Y^\circ, Y) = s(Z, Y)$.*

(The equality of supports allows us to identify the relevant Chow groups, as required in order to compare the Segre classes, cf. Remark 2.2.)

Thus, the Segre class of Z in Y can be 'cut out' by $\dim Y + 1$ hypersurfaces, and by $\dim Y$ hypersurfaces in a neighborhood of Z. This fact is reminiscent of a well-known result of D. Eisenbud and G. Evans [4], stating that every subscheme Z of $\mathbb{P}^n$ may be cut out set-theoretically by n hypersurfaces, and of an observation by W. Fulton [6, Example 9.1.3] pointing out that $n+1$ hypersurfaces suffice to cut out Z *scheme-theoretically* if Z is locally a complete intersection. As a particular case of Theorem 1.1, $n+1$ hypersurfaces suffice to cut out a subscheme $Z' \subseteq \mathbb{P}^n$ with the same Segre class in $\mathbb{P}^n$ as Z, without any requirement on Z. These hypersurfaces may be chosen to be general in a linear system cutting out Z, and n hypersurfaces suffice in a neighborhood of Z.

Theorem 1.1 may be further refined, as follows. Denote by $s(Z, Y)_k$ the k-dimensional component of the Segre class $s(Z, Y)$.

Theorem 1.1. (Continued.)

(b') *More generally, let $c \geq 0$ and let $Z_{(c)} := X_1 \cap \cdots \cap X_{\dim Y-c}$. Then there exists a closed subscheme S of dimension $\leq c$ in Y such that* $\dim(S \cap Z) < c$, *$(Z_{(c)} \smallsetminus S)_{red} = (Z \smallsetminus S)_{red}$, $s(Z_{(c)} \smallsetminus S, Y \smallsetminus S)_c = s(Z \smallsetminus S, Y \smallsetminus S)_c$, and $s(Z_{(c)}, Y)_k = s(Z, Y)_k$ for $k > c$.*

For $c = 0$, part (b') of Theorem 1.1 reduces to part (b): in this case S is a set of points disjoint from Z, and we can take $Y^\circ = Y \smallsetminus S$. Part ($a$) may also formally be seen as a particular case of (b'), by allowing $c = -1$ (and hence $S = \emptyset$).

If V is an irreducible component of Z, it follows from Theorem 1.1(b') with $c = \dim V$ that the coefficient of V in $s(Z, Y)$ equals the coefficient of V in $s(Z_{(\dim V)}, Y)$. As this coefficient equals the multiplicity of Y along Z at V [6, Example 4.3.4], this recovers Samuel's result.