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A note on perfect isometries between finite general linear and unitary groups at unitary primes

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ABSTRACT

Let q be a power of a prime, ℓ a prime not dividing q, d a positive integer coprime to both ℓ and the multiplicative order of $q \mod \ell$ and n a positive integer. A. Watanabe proved that there is a perfect isometry between the principal ℓ-blocks of $\mathrm{GL}_n(q)$ and $\mathrm{GL}_n(q^d)$ where the correspondence of characters is given by Shintani descent. In the same paper Watanabe also proved that if ℓ and q are odd and ℓ does not divide $|\mathrm{GL}_n(q^2)|/|\mathrm{U}_n(q)|$ then there is a perfect isometry between the principal ℓ-blocks of $\mathrm{U}_n(q)$ and $\mathrm{GL}_n(q^2)$ with the correspondence of characters also given by Shintani descent. R. Kessar extended this first result to all unipotent blocks of $\mathrm{GL}_n(q)$ and $\mathrm{GL}_n(q^d)$. In this paper we extend this second result to all unipotent blocks of $\mathrm{U}_n(q)$ and $\mathrm{GL}_n(q^2)$. In particular this proves that any two unipotent blocks of $\mathrm{U}_n(q)$ at unitary primes (for possibly different n) with the same weight are perfectly isometric. We also prove that this perfect isometry commutes with Deligne–Lusztig induction at the level of characters.

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1. Preliminaries

1.1. Perfect isometries

Let ℓ be an odd prime and consider an ℓ-modular system $(K, \mathcal{O}, k)$, such that K contains enough roots of unity for the groups being considered in this paper. Let G and H be finite groups and b and c block idempotents of $\mathcal{O}G$ and $\mathcal{O}H$ respectively. We denote by $\mathrm{Irr}(G, b)$ the set of irreducible characters of $\mathcal{O}Gb$. A perfect isometry (see [2]) between $\mathcal{O}Gb$ and $\mathcal{O}Hc$ is an isometry

$$I : \mathbb{Z}\,\mathrm{Irr}(G,b) \xrightarrow{\sim} \mathbb{Z}\,\mathrm{Irr}(H,c),$$
$$\text{with } I(\mathbb{Z}\,\mathrm{Irr}(G,b)) = \mathbb{Z}\,\mathrm{Irr}(H,c),$$

where

$$\hat{I} : G \times H \to \mathcal{O}$$
$$(x,y) \mapsto \sum_{\chi \in \mathrm{Irr}(G,b)} \chi(x) I(\chi)(y)$$

has the property that $\hat{I}(x,y)$ is divisible by $|C_G(x)|$ and $|C_G(y)|$ in $\mathcal{O}$ for all $x \in G$ and $y \in H$ and that $\hat{I}(x,y) = 0$ if in addition exactly one of x and y is ℓ-singular.

1.2. Shintani descent

Let n be a positive integer and q a power of an odd prime p. We denote by F the Frobenius endomorphism of $\mathbf{G} := \mathrm{GL}_n(\overline{\mathbb{F}_q})$ defined by

$$F : \mathbf{G} \to \mathbf{G}$$
$$A \mapsto JA^{-t[q]}J,$$

where $A^{[q]}$ is the matrix obtained from A by raising every entry to the power q, A^{-t} is the inverse transpose of A and J is the $n \times n$ matrix with 1's on the anti-diagonal and 0's everywhere else. We define

$$\mathbf{G}^F =: \mathrm{U}_n(q)$$
$$\mathbf{G}^{F^2} =: \mathrm{GL}_n(q^2).$$

We denote by σ the restriction of F to $\mathbf{G}^{F^2}$. Two elements $g, h \in \mathbf{G}^{F^2}$ are said to be σ-conjugate if $xg\sigma(x)^{-1} = h$ for some $x \in \mathbf{G}^{F^2}$. For $g \in \mathbf{G}$, we have $g^{-1}F(g) \in \mathbf{G}^{F^2}$ iff $F^2(g)g^{-1} \in \mathbf{G}^F$. Now by Lang's theorem, any element x of $\mathbf{G}^F$ can be written as $x = g^{-1}F(g)$ for some $g \in \mathbf{G}$. The map $g^{-1}F(g) \to F^2(g)g^{-1}$ gives a bijection, that