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The bi-graded structure of symmetric algebras with applications to Rees rings [☆]

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A B S T R A C T

Consider a rational projective plane curve $\mathcal{C}$ parameterized by three homogeneous forms of the same degree in the polynomial ring $R = k[x, y]$ over a field k. The ideal I generated by these forms is presented by a homogeneous 3×2 matrix φ with column degrees $d_1 \le d_2$. The Rees algebra $\mathcal{R} = R[It]$ of I is the bi-homogeneous coordinate ring of the graph of the parameterization of $\mathcal{C}$; and accordingly, there is a dictionary that translates between the singularities of $\mathcal{C}$ and algebraic properties of the ring $\mathcal{R}$ and its defining ideal. Finding the defining equations of Rees rings is a classical problem in elimination theory that amounts to determining the kernel $\mathcal{A}$ of the natural map from the symmetric algebra $\operatorname{Sym}(I)$ onto $\mathcal{R}$. The ideal $\mathcal{A}_{\ge d_2-1}$, which is an approximation of $\mathcal{A}$, can be obtained using linkage. We exploit the bi-graded structure of $\operatorname{Sym}(I)$ in order to describe the structure of an improved approximation $\mathcal{A}_{\ge d_1-1}$ when $d_1 < d_2$ and φ has a generalized zero in its first column. (The latter condition is equivalent to assuming that $\mathcal{C}$ has a singularity of multiplicity d_2.) In particular, we give the bi-degrees of a minimal bi-homogeneous generating set for

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Morley forms
Parametrization
Rational plane curve
Rational plane sextic
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Symmetric algebra

this ideal. When $2 = d_1 < d_2$ and φ has a generalized zero in its first column, then we record explicit generators for $\mathcal{A}$. When $d_1 = d_2$, we provide a translation between the bi-degrees of a bi-homogeneous minimal generating set for $\mathcal{A}_{d_1-2}$ and the number of singularities of multiplicity d_1 that are on or infinitely near $\mathcal{C}$. We conclude with a table that translates between the bi-degrees of a bi-homogeneous minimal generating set for $\mathcal{A}$ and the configuration of singularities of $\mathcal{C}$ when the curve $\mathcal{C}$ has degree six.

Contents

1. Introduction

Our basic setting is as follows: Let k be an algebraically closed field, $R = k[x, y]$ a polynomial ring in two variables, and I an ideal of R minimally generated by homogeneous forms h_1, h_2, h_3 of the same degree $d > 0$. Extracting a common divisor we may harmlessly assume that I has height two. We will keep these assumptions throughout the introduction, though many of our results are stated and proved in greater generality.

On the one hand, the homogeneous forms h_1, h_2, h_3 define a morphism

$$\eta : \mathbb{P}^1_k \xrightarrow{[h_1:h_2:h_3]} \mathbb{P}^2_k \tag{1.0.1}$$

whose image is a curve $\mathcal{C}$. After reparameterizing we may assume that the map η is birational onto its image or, equivalently, that the curve $\mathcal{C}$ has degree d.

On the other hand, associated to h_1, h_2, h_3 is a syzygy matrix φ that gives rise to a homogeneous free resolution of the ideal I,

$$0 \longrightarrow R(-d-d_1) \oplus R(-d-d_2) \xrightarrow{\varphi} R(-d)^3 \longrightarrow I \longrightarrow 0\ .$$