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# On the projective normality of double coverings over a rational surface

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ABSTRACT

We study the projective normality of a minimal surface $X$ which is a ramified double covering over a rational surface $S$ with $\dim|-K_S| \geq 1$. In particular Horikawa surfaces, the minimal surfaces of general type with $K_X^2 = 2p_g(X) - 4$, are of this type, up to resolution of singularities. Let $\pi$ be the covering map from $X$ to $S$. We show that the $\mathbb{Z}_2$-invariant adjoint divisors $K_X + r\pi^*A$ are normally generated, where the integer $r \geq 3$ and $A$ is an ample divisor on $S$.
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## 1. Introduction

Let $X$ be a smooth complex projective variety and $L$ be a very ample line bundle on $X$, inducing a closed embedding

$$\varphi : X \hookrightarrow \mathbb{P}(V),$$

where $V = H^0(X, L)$. A natural question concerning such an embedding is that for which $L$, is the natural map

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$$H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(k)) \to H^0(X, L^{\otimes k})$$

surjective for every positive integer $k$? Put it another way, for which $L$, can every member $D \in |L^{\otimes k}|$ be cut out from $X$ by a degree $k$ hypersurface in $\mathbb{P}(V)$?

If the answer to the above question is positive, then $X$ is embedded by the complete linear system $|L|$ as a projectively normal variety and $L$ is said to be *normally generated*.

Normal generation is equivalent to so-called Property $(N_0)$. More generally, one can define Property $(N_p)$ for any integer $p \geq 0$. These properties prescribe the shape of a minimal graded free resolution of $R(L) := \oplus_{k\geq 0} H^0(X, L^{\otimes k})$ as an $S := \mathrm{Sym} H^0(X, L)$-module. We refer the reader to [14] for an account of this subject.

A conjecture attributed to S. Mukai says, using the additive language of divisors, that for a smooth projective variety $X$ of dimension $n$, divisors of the form $K_X + (n + 2 + p)A + P$ shall satisfy Property $(N_p)$, where $K_X$ is the canonical divisor of $X$, $A$ is an ample divisor on $X$ and $P$ is a nef divisor. This was confirmed, in a stronger form, in the case that $A$ is very ample by [4]; and in the end of that paper, among other questions, the following was raised.

**Conjecture 1.1.** *If $X$ is a smooth projective surface, and $A$ is an ample divisor on $X$, then $K_X + rA$ is normally generated for every integer $r \geq 4$.*

It is well known that if $r \geq 4$, $K_X + rA$ is very ample by [17]. Concerning normal generation, Conjecture 1.1 has been known to be true in several important cases: $K3$ surfaces by [15,18], Abelian surfaces by [13], elliptic ruled surfaces by [10,11], and anti-canonical rational surfaces by [6]. For minimal surfaces of general type, partial results are obtained, for instance, if $K_X^2 \geq 2$ and $A - K_X$ is big and nef, then $K_X + rA$ is normally generated for $r \geq 2$ by [16].

In this note, our main result is

**Theorem 1.2.** *Let $S$ be a rational surface with $\dim|-K_S| \geq 1$. Let $\pi : X \to S$ be a ramified double covering of $S$ by a minimal surface $X$ (possibly singular). Let $L$ be a divisor on $S$ with the property that $K_S + L$ is nef and $L \cdot C \geq 3$ for any curve $C$. Then $K_X + \pi^* L$ is base point free and the natural map*

$$Sym^r H^0(K_X + \pi^* L) \to H^0(r(K_X + \pi^* L)) \tag{1.1}$$

*surjects for every $r \geq 1$.*

Typical examples for $X$ include Horikawa surfaces (see Section 2) and the $K3$ surfaces obtained by taking double cover of $\mathbb{P}^2$ branched along a smooth sextic. We remark that the condition that $K_S + L$ is nef and $L \cdot C \geq 3$ for any curve $C$ amounts to that $L^2 \geq 7$ and $L \cdot C \geq 3$ for any curve $C$, see Proposition 3.8. Thus the following is an immediate corollary of Theorem 1.2, which further presents some evidence for Conjecture 1.1.

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