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Principal minor ideals and rank restrictions on their vanishing sets

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ABSTRACT

All matrices we consider have entries in a fixed algebraically closed field K. A minor of a square matrix is *principal* means it is defined by the same row and column indices. We study the ideal generated by size t principal minors of a generic matrix, and restrict our attention to locally closed subsets of its vanishing set, given by matrices of a fixed rank. The main result is a computation of the dimension of the locally closed set of $n \times n$ rank $n-2$ matrices whose size $n-2$ principal minors vanish; this set has dimension $n^2 - n - 4$.

1. Introduction

Given a generic $n \times n$ matrix X, and $K[X]$ the polynomial ring in entries x_{ij} of X over some algebraically closed field K, we study the ideals $\mathfrak{P}_t = \mathfrak{P}_t(X)$, generated by the size t principal minors of X. Historically, various ideals defined using generic matrices have been of great interest to algebraistis – such examples include the determinantal ideals (see [4–6,9,19,23]), due to their connection to invariant theory (as in [2]) and the Pfaffian ideals (see [3,10–12,18]), whose study is often inspired by the result from [1],

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as well as their connection to invariant theory. In developing their generalized version of the Principal Minor Theorem, Kodiyalam, Lam, and Swan [14] reveal a contrast between the principal minor ideals and the Pfaffian ideals: while the Pfaffian ideals, like the determinantal ideals, satisfy a chain condition according to rank, the principal minor ideals do not. Furthermore, in [24] it is shown that, unlike determinantal ideals and Pfaffian ideals, principal minor ideals are not, in general, Cohen–Macaulay.

Principal minors arise in many other contexts – see, for example, [8,15,16,21,22]. The most direct study of principal minors is in [24]. There, it is shown the algebraic set $\mathcal{V}(\mathfrak{B}_{n-1})$ has two components: one given by the determinantal ideal I_{n-1} and the other given by a height n ideal, $\mathfrak{Q}_{n-1}$, that is the contraction to $K[X]$ of the kernel of the map

$$K[X]\left[\frac{1}{\det X}\right] \to K[X]\left[\frac{1}{\det X}\right]/\mathfrak{B}_1$$
$$X \to X^{-1}.$$

When $n = 4$, $\mathfrak{B}_{n-1}$ is reduced and as a consequence, I_{n-1} and $\mathfrak{Q}_{n-1}$ are linked in that case. Identifying the components for $\mathcal{V}(\mathfrak{B}_{n-1})$ relies on another result within that paper, that if $\mathcal{Y}_{n,r,t}$ denotes the locally closed set of $\mathcal{V}(\mathfrak{B}_t)$ consisting of rank r matrices, then for all n, t,

$$\mathcal{Y}_{n,n,t} \cong \mathcal{Y}_{n,n,n-t}$$

as schemes.

This paper is organized as follows: Section 2 gives the necessary preliminaries for the remainder of the paper. We focus on the components of $\mathcal{V}(\mathfrak{B}_t)$ by restricting to the locally closed subsets $\mathcal{Y}_{n,r,t}$, consisting of matrices of rank exactly r. Our main result, given in Section 3, is a computation of the dimension of $\mathcal{Y}_{n,n-2,n-2}$.

Theorem *(3, Section 3.3). The locally closed set of* $n\times n$ *rank* $n-2$ *matrices in* $\operatorname{Spec} K[X]$, *whose size* $n-2$ *principal minors vanish, has dimension* $n^2 - 4 - n$.

In studying the components of $\mathcal{Y}_{n,n-2,n-2}$ we define a bundle map Θ (see Equation (1), Section 2) that reduces the problem to studying pairs of subsets in $\mathrm{Grass}(n-2, n)$. The technique in proving Theorem 3 is as follows: Given a point in the Grassmannian, we encode exactly which of its Plücker coordinates do and do not vanish in a simple graph. Such graphs are called *permissible* (see Section 3.2). We then define the notion of a permissible subvariety of the Grassmannian, along with its corresponding graph. We prove and then use the properties of permissible graphs to compute the dimension of $\mathcal{Y}_{n,n-2,n-2}$.

In Section 3.4 we suggest a natural extension of the techniques from Section 3.2 to the locally closed sets $\mathcal{Y}_{n,n-3,n-3}$. More generally, the structure of $\mathcal{Y}_{n,t,t}$ turns out to be of great interest in its own right, in fact leading to questions that are NP-hard (see [7]