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On functional equations of finite multiple polylogarithms

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Recently, several people study finite multiple zeta values (FMZVs) and finite polylogarithms (FPs). In this paper, we introduce finite multiple polylogarithms (FMPs), which are natural generalizations of FMZVs and FPs, and we establish functional equations of FMPs. As applications of these functional equations, we calculate special values of FMPs containing generalizations of congruences obtained by Meštrović, Z.W. Sun, L.L. Zhao, Tauraso, and J. Zhao.

1. Introduction

From the end of twentieth century to the beginning of twenty-first century, Hoffman and J. Zhao had started research about mod p multiple harmonic sums, which are motivated by various generalizations of classical Wolstenholme's theorem. Recently, Kaneko and Zagier introduced a new "adélic" framework to describe the pioneer works by

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Hoffman and Zhao and they defined finite multiple zeta values (FMZVs). Let $k_1, \dots, k_m$ be positive integers and $\Bbbk := (k_1, \dots, k_m)$.

Definition 1.1 *(Kaneko and Zagier [8,9]). The finite multiple zeta value* $\zeta_{\mathcal{A}}(\Bbbk)$ is defined by

$$\zeta_{\mathcal{A}}(\Bbbk) := \left(\sum_{p>n_1>\cdots>n_m>0} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}$$

and *the finite multiple zeta-star value* $\zeta^{\star}_{\mathcal{A}}(\Bbbk)$ is defined by

$$\zeta^{\star}_{\mathcal{A}}(\Bbbk) := \left(\sum_{p-1\geq n_1\geq\cdots\geq n_m\geq 1} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}.$$

Here, the $\mathbb{Q}$-algebra $\mathcal{A}$ is defined by

$$\mathcal{A} := \left(\prod_p \mathbb{F}_p \right) \Big/ \left(\bigoplus_p \mathbb{F}_p \right),$$

where p runs over all prime numbers.

In this framework, Kaneko and Zagier established a conjecture, which states that there is an isomorphism between the $\mathbb{Q}$-algebra spanned by FMZVs and the quotient $\mathbb{Q}$-algebra modulo the ideal generated by $\zeta(2)$ of the $\mathbb{Q}$-algebra spanned by the usual multiple zeta values.

On the other hand, Kontsevich [12], Elbaz-Vincent and Gangl [2] introduced finite version of polylogarithms and studied functional equations of them. Based on their works, Mattarei and Tauraso [16] calculated special values of finite polylogarithms.

Inspired by these studies, we introduce a finite version of multiple polylogarithms in the framework of Kaneko and Zagier:

Definition 1.2 *(See Definition 3.8). The finite multiple polylogarithms* (FMPs) $\pounds_{\mathcal{A},\Bbbk}(t)$, $\pounds^{\star}_{\mathcal{A},\Bbbk}(t)$, $\widetilde{\pounds}_{\mathcal{A},\Bbbk}(t)$, and $\widetilde{\pounds}^{\star}_{\mathcal{A},\Bbbk}(t)$ are defined by

$$\pounds_{\mathcal{A},\Bbbk}(t) := \left(\sum_{p>n_1>\cdots>n_m>0} \frac{t^{n_1}}{n_1^{k_1}\cdots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}_{\mathbb{Z}[t]},$$

$$\pounds^{\star}_{\mathcal{A},\Bbbk}(t) := \left(\sum_{p-1\geq n_1\geq\cdots\geq n_m\geq 1} \frac{t^{n_1}}{n_1^{k_1}\cdots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}_{\mathbb{Z}[t]},$$

$$\widetilde{\pounds}_{\mathcal{A},\Bbbk}(t) := \left(\sum_{p>n_1>\cdots>n_m>0} \frac{t^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}} \bmod p \right)_p \in \mathcal{A}_{\mathbb{Z}[t]},$$