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The Hilbert–Kunz functions of two-dimensional rings of type ADE

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A B S T R A C T

We compute the Hilbert–Kunz functions of two-dimensional rings of type ADE by using representations of their indecomposable, maximal Cohen–Macaulay modules in terms of matrix factorizations, and as first syzygy modules of homogeneous ideals.

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Introduction

The central objects in this paper will be the two-dimensional *rings of type ADE* (or *ADE*-rings for short), namely

- $A_n := k[X,Y,Z]/(X^{n+1} - YZ)$ with $n \geq 0$,
- $D_n := k[X,Y,Z]/(X^2 + Y^{n-1} + YZ^2)$ with $n \geq 4$,
- $E_6 := k[X,Y,Z]/(X^2 + Y^3 + Z^4)$,
- $E_7 := k[X,Y,Z]/(X^2 + Y^3 + YZ^3)$ and
- $E_8 := k[X,Y,Z]/(X^2 + Y^3 + Z^5)$,

as well as their (X,Y,Z)-adic completions, where k denotes an algebraically closed field. These rings were studied by several authors, e.g. Klein (in [18]), du Val (in [11]), Brieskorn (in [9]) or Artin (in [1,2]), where they appeared in various forms, e.g as quotient singularities or rational double points. They also appear in string theory (cf. [26] for a survey).

The goal is to compute (in positive characteristic) their *Hilbert–Kunz functions*

$$e \mapsto \dim_k \left(k[X,Y,Z]/\left(F, X^{p^e}, Y^{p^e}, Z^{p^e}\right)\right),$$

where F denotes one of the defining polynomials above. Note that the (non-local) rings of type ADE are homogeneous with respect to some positive grading.

Recall that the rings $\mathbb{C}[X,Y,Z]/(F)$ of type ADE appear as rings of invariants of $\mathbb{C}[x,y]$ by the actions of the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. The groups corresponding to the singularities of type A_n, D_n, E_6, E_7 resp. E_8 are the cyclic group with $n+1$ elements, the binary dihedral group $\mathbb{D}_{n-2}$ of order $4n-8$, the binary tetrahedral group $\mathbb{T}$ of order 24, the binary octahedral group $\mathbb{O}$ of order 48 resp. the binary icosahedral group $\mathbb{I}$ of order 120. If k is algebraically closed of characteristic $p > 0$ the groups above can be viewed as finite subgroups of $\mathrm{SL}_2(k)$, provided their order is invertible in k. In these cases $k[X,Y,Z]/(F)$ is again the ring of invariants of $k[x,y]$ under the action of the corresponding group (cf. [22, Chapter 6, §2]). Using this fact, Watanabe and Yoshida showed in [27] that the Hilbert–Kunz multiplicity of rings of type ADE is given by $2 - \frac{1}{|G|}$, provided the order of the corresponding group G is invertible modulo p. Moreover, by a result of Brenner (cf. [8]) the Hilbert–Kunz functions are of the form

$$e \mapsto \mathrm{e}_{\mathrm{HK}} \cdot p^{2e} + \gamma(p^e),$$

where γ is an eventually periodic function. For example, the Hilbert–Kunz function of a ring of type A_n is due to Kunz (cf. [21, Example 4.3]) given by

$$e \mapsto \left(2 - \frac{1}{n+1}\right) p^{2e} - r + \frac{r^2}{n+1},$$