Journal of Algebra 469 (2017) 421–436

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Polynomial codimension growth and the Specht problem

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A R T I C L E   I N F O

Article history:
Received 3 February 2016
Available online 15 September 2016
Communicated by Louis Rowen

MSC:
primary 17A30, 16R10
secondary 16P90

Keywords:
Polynomial identity
Codimension
Growth

A B S T R A C T

We construct a continuous family of algebras over a field of characteristic zero with slow codimension growth bounded by a polynomial of degree 4. This is achieved by building, for any real number $\alpha \in (0,1)$ a commutative nonassociative algebra A_α whose codimension sequence $c_n(A_\alpha)$, $n = 1, 2, \ldots$, is polynomially bounded and $\lim \log_n c_n(A_\alpha) = 3 + \alpha$.
As an application we are able to construct a new example of a variety with an infinite basis of identities.

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[1] Authors were partially supported by the GNSAGA of INdAM.
[2] Author was supported by the Russian Science Foundation, Grant No. 16-11-10013.

http://dx.doi.org/10.1016/j.jalgebra.2016.09.008
0021-8693/

1. Introduction

Let F be a field of characteristic zero and $F\{X\}$ the free nonassociative algebra on a countable set X over F. If A is a nonnecessarily associative algebra over F we denote by $Id(A)$ the T-ideal of polynomial identities of A. In the study of $Id(A)$ an important role is played by the sequence $c_n(A)$, $n = 1, 2, \ldots$, of codimensions of A which is obtained as an evaluation of a sequence of characters of the symmetric group. In fact a general strategy in the study of $Id(A)$ is that of studying the space of multilinear polynomials in n fixed variables modulo the identities of the algebra A, through the representation theory of the symmetric group on n symbols. Then one attaches to $Id(A)$ a sequence of S_n-modules, $n = 1, 2, \ldots$, and studies the corresponding sequence of characters.

The sequence of codimensions is completely determined by the polynomial identities of A, and one can find in [6] the basic notions and properties. The growth function determined by the sequence of integers $c_n(A)$, $n = 1, 2, \ldots$ is the codimension growth of the algebra A. Now, in general, the codimensions are bounded only by an overexponential function

$$c_n(A) \leq \frac{1}{n}\binom{2n-2}{n-1} n!$$

where $\frac{1}{n}\binom{2n-2}{n-1}$ is the n-th Catalan number. Nevertheless for an arbitrary associative algebra A we have that $c_n(A) \leq n!$ whereas for any Lie algebra $c_n(A) \leq (n-1)!$. The sequence of codimensions can be further bounded. In fact, if A is an associative PI-algebra then $c_n(A)$ is exponentially bounded, i.e., $c_n(A) \leq a^n$ for some real a, and if A is an arbitrary (nonnecessarily associative) finite dimensional algebra then $c_n(A) \leq (\dim A)^{n+1}$ (see [1,9]).

We say that an algebra A has polynomial growth of the codimensions if there exist real numbers α and t such that

$$c_n(A) \leq \alpha n^t, \tag{1}$$

for all $n \geq 1$. It is worth noticing that in several important cases the polynomial codimension growth has a very concrete behavior. For example, if A is an associative PI-algebra then (1) implies that

$$c_n(A) = qn^t + O(n^{t-1}) \tag{2}$$

where t is a positive integer and q is a positive real number (see [3]). Also, if A is a unitary algebra then q is a rational number such that

$$\frac{1}{t!} \leq q \leq \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^t}{t!} \approx \frac{1}{e}$$