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Torsion and divisibility for reciprocity sheaves and 0-cycles with modulus

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ABSTRACT

The notion of *modulus* is a striking feature of Rosenlicht–Serre's theory of generalized Jacobian varieties of curves. It was carried over to algebraic cycles on general varieties by Bloch–Esnault, Park, Rülling, Krishna–Levine. Recently, Kerz–Saito introduced a notion of Chow group of 0-cycles with modulus in connection with geometric class field theory with wild ramification for varieties over finite fields. We study the non-homotopy invariant part of the Chow group of 0-cycles with modulus and show their torsion and divisibility properties.
Modulus is being brought to sheaf theory by Kahn–Saito–Yamazaki in their attempt to construct a generalization of Voevodsky–Suslin–Friedlander's theory of homotopy invariant presheaves with transfers. We prove parallel results about torsion and divisibility properties for them.

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1. Introduction

Let k be a field and let $\overline{X}$ be a proper k-variety equipped with an effective Cartier divisor D. For such a pair $(\overline{X}, D)$, Kerz and Saito recently defined in [7] a notion of Chow group $\mathrm{CH}_0(\overline{X}|D)$ of 0-cycles on $\overline{X}$ with modulus D as a quotient of the group $Z_0(X)$ of 0-cycles on the open complement $X := \overline{X} \setminus |D|$.

The Kerz–Saito Chow group of 0-cycles with modulus is one of the most recent developments of the emerging theory of algebraic cycles with certain constraints at infinity. The idea originated from the work of Bloch and Esnault [4], and was subsequently developed in [15,17,9–11].

When $\overline{X}$ is a smooth projective curve, the group $\mathrm{CH}_0(\overline{X}|D)$ is isomorphic to the relative Picard group $\mathrm{Pic}(\overline{X}, D)$ of isomorphism classes of pairs given by a line bundle on $\overline{X}$ together a trivialization along D. Its degree-0-part agrees with the group of k-rational points of the generalized Jacobian $\mathrm{Jac}(\overline{X}|D)$ of Rosenlicht and Serre (see, for instance, [19, Chapter II]). If D is non-reduced, then $\mathrm{Jac}(\overline{X}|D)$ is a commutative algebraic group of general type, i.e. an extension of a semi-abelian variety by a unipotent group, which depends on the multiplicity of D. The existence of such non-homotopy invariant part suggests that the group $\mathrm{CH}_0(\overline{X}|D)$ may give new geometric and arithmetic information about the pair $(\overline{X}, D)$ that cannot be captured by the classical (homotopy invariant) motivic cohomology groups.

Intimately connected with the world of cycles subject to some modulus conditions is the recent work of Kahn, Saito and Yamazaki [6], which gives a categorical attempt at the quest for a non-homotopy-invariant motivic theory. This encompasses unipotent phenomena and is modeled on the generalized Jacobians of Rosenlicht and Serre. Kahn–Saito–Yamazaki developed the notion of "reciprocity" for (pre)sheaves with transfers, which is weaker than homotopy invariance, with the purpose of eventually constructing a new motivic triangulated category, larger than Voevodsky's $\mathbf{DM}^{\mathrm{eff}}(k, \mathbb{Z})$ and containing unipotent information.

The goal of this paper is to exhibit some differences between the classical homotopy invariant objects and the new non-homotopy invariant ones, such as 0-cycles with modulus and reciprocity sheaves.

For 0-cycles, we shall see in §2.2 that there is a canonical surjection from the Chow group with modulus to the 0-th Suslin homology group (as defined e.g. in [14, Definition 7.1])

$$\pi_{\overline{X},D} \colon \mathrm{CH}_0(\overline{X}|D) \longrightarrow \mathrm{H}_0^{\mathrm{Sing}}(X).$$

Since $\mathrm{H}_0^{\mathrm{Sing}}(X)$ is the maximal homotopy invariant quotient of the group $Z_0(X)$ of 0-cycles on X, the kernel $\mathrm{U}(\overline{X}|D)$ of $\pi_{\overline{X},D}$ measures the failure of $\mathrm{CH}_0(\overline{X}|D)$ to be homotopy invariant (nonetheless, its degree-0-part enjoys $\mathbb{P}^1$-invariance as pointed out in Remark 2.4). The first result of this paper is the following divisibility property of $\mathrm{U}(\overline{X}|D)$: