Journal of Algebra 468 (2016) 72–79

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the focal subgroup of a saturated fusion system

Adolfo Ballester-Bolinches [a], Luis M. Ezquerro [b], Ning Su [c,*], Yanming Wang [d]

[a] *Departament d'Àlgebra, Universitat de València, Dr. Moliner, 50, E-46100 Burjassot, València, Spain*
[b] *Departamento de Matemáticas, INAMAT, Universidad Pública de Navarra, Campus de Arrosadía, E-31006 Pamplona, Navarra, Spain*
[c] *Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, China*
[d] *Lingnan College and Dept. of Math., Sun Yat-sen University, Guangzhou, 510275, China*

ARTICLE INFO

Article history:
Received 12 April 2016
Available online 18 August 2016
Communicated by Markus Linckelmann

MSC:
20C20

Keywords:
Saturated fusion system
Control of fusion

ABSTRACT

The influence of the cyclic subgroups of order p or 4 of the focal subgroup of a saturated fusion system $\mathcal{F}$ over a p-group S is investigated in this paper. Some criteria for normality of S in $\mathcal{F}$ as well as necessary and sufficient conditions for nilpotency of $\mathcal{F}$ are given. The resistance of a p-group in which every cyclic subgroup of order p or 4 is normal, and earlier results about p-nilpotence of finite groups and nilpotency of saturated fusion systems are consequences of our study.

1. Introduction and statements of results

All groups considered in this paper are finite and p is always understood to be a fixed prime. We shall adhere to the notation used in [1] and we refer the reader to that book for the basic notation, terminology and results.

* Corresponding author.
E-mail addresses: Adolfo.Ballester@uv.es (A. Ballester-Bolinches), ezquerro@unavarra.es (L.M. Ezquerro), suning3@mail.sysu.edu.cn (N. Su), stswym@mail.sysu.edu.cn (Y. Wang).

http://dx.doi.org/10.1016/j.jalgebra.2016.07.034

The theory of fusion systems is becoming an important topic in algebra, with emerging interactions with local finite group theory, representation theory and algebraic topology. Recall that if S is a Sylow p-subgroup of a group G, then the *fusion system of G on S* is the category $\mathcal{F}_S(G)$, whose objects are all subgroups of S and whose morphisms are those induced by conjugation in G (see [1, Part I, Definition 1.1]). $\mathcal{F}_S(G)$, that was originally considered by Puig in the seventies, describes how subgroups of S are related by G-conjugation and determines a large part of the p-local behaviour of the group G. In fact many classical results of fusion in Sylow p-subgroups as Burnside and Frobenius p-nilpotence criteria, Alperin's fusion Theorem and Goldschmidt's fusion Theorem can be interpreted as results about $\mathcal{F}_S(G)$.

The above category can be considered as a standard example of a saturated fusion system, a concept introduced by Puig in the early nineties. A *saturated fusion system over a p-group S* is a category whose objects are the subgroups of S, and whose morphisms satisfy certain axioms mimicking those satisfied by morphisms induced by conjugation in some ambient group over the subgroups of one of its Sylow p-subgroups (see [1, Part I, Definition 1.1]).

This article is concerned with the focal subgroup $\mathfrak{foc}_G(S)$ of a Sylow p-subgroup S of a group G and its extension within the framework of fusion systems. The importance of the focal subgroup in this context comes from the fact that it is determined entirely by p-fusion and detects whether the whole group has a non-trivial p-factor group. According to [6, Theorem 7.3.4], we have the following description of the focal subgroup.

$$\mathfrak{foc}_G(S) = S \cap G' = \langle x^{-1}x^g : x \in S, g \in G \text{ such that } x^g \in S \rangle.$$

This description seems to be appropriated as a definition of the focal subgroup of the fusion system $\mathcal{F}_G(S)$ and motivates the definition of the focal subgroup of a saturated fusion system:

Definition 1.1 *([1, Part I, Definition 7.1]).* For any saturated fusion system $\mathcal{F}$ on a p-group S, the *focal subgroup*, $\mathfrak{foc}(\mathcal{F})$, is

$$\mathfrak{foc}(\mathcal{F}) = \langle [P, \mathrm{Aut}_{\mathcal{F}}(P)] \colon P \leq S \rangle = \langle g^{-1}g^{\alpha} \colon g \in P \leq S, \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \rangle.$$

Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Recall that S is *normal* in $\mathcal{F}$ if for all $P \leq S$ and all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, φ extends to a morphism $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(S)$ (see [1, Part I, Definition 4.1]). If S is normal in any saturated fusion system over S, we say that S is *resistant* [9].

Our main result gives a necessary and sufficient condition for a p-group S to be normal in a saturated fusion system $\mathcal{F}$ over S in terms of the subgroups of prime order or order 4 of $\mathfrak{foc}(\mathcal{F})$.

Theorem A. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Then S is normal in $\mathcal{F}$ if and only if for each cyclic subgroup P of $\mathfrak{foc}(\mathcal{F})$ of order p or 4 and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \mathfrak{foc}(\mathcal{F}))$, φ extends to a morphism $\bar{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(S)$.*