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Hereditary arithmetics ☆

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ABSTRACT

The abstract arithmetic of non-commutative non-singular arithmetic curves (equivalently: the ideal theory of hereditary orders) is revisited in the framework of quantum B-algebras. It is shown that multiplication of ideals can be transformed into composition of functions. This yields a non-commutative "fundamental theorem of arithmetic" extending the classical one. Local hereditary arithmetics are presented by generators and relations and correlated with tubular quantum B-algebras. Main results are achieved by a divisor theory which furnishes the divisor group with a ring-like structure satisfying a 1-cocycle condition.

1. Introduction

Is there a non-commutative fundamental theorem of arithmetic? Taking Dedekind's unique factorization into prime ideals as the commutative paradigm, the question aims at a precise analogue in the framework of non-commutative arithmetic in dimension one.

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In a first approximation, the latter means that instead of a Dedekind domain R with quotient field K, we have to think of an *R-order* Λ in a finite dimensional K-algebra A, that is, a module-finite R-subalgebra Λ of A with $K\Lambda = A$. According to the Dedekind property of R, we have to assume that Λ is *hereditary*, that is, of global dimension one (see [27], chapter 9).

Note that hereditary arithmetics, the pattern behind the ideal theory of hereditary orders, occurs in various contexts like coherent sheaves on a weighted projective line [14,22], or in connection with the tubular structure of Auslander–Reiten components of tame algebras [10,28] (see Remark 2 of Section 7).

For maximal orders Λ, a strict analogue of Dedekind's fundamental theorem, even without the assumption that Λ is finite over its centre, was proved in 1939 by Asano [2]. Here the arithmetic is the same as in the commutative case. If Λ is hereditary, but not maximal, there are finite cliques of non-commuting prime ideals, every fractional ideal is projective as a left and as a right Λ-module, but no longer invertible. The ideal theory of hereditary orders was studied since 1963 by Harada [15–18], Brumer [8], and Jacobinski [20], later on in the more general context of hereditary noetherian prime rings by Eisenbud and Robson [12] and others (see e.g., the monograph [23]). It was shown, for example, that each ideal is a product of an invertible ideal and an ideal some power of which is idempotent ([12], Theorem 4.2). A precise description, however, comparable with the fundamental theorem in the commutative case, never appeared.

In the present paper, this gap will be closed. We establish a bijection between fractional ideals and non-decreasing periodic functions, so that the multiplication changes into composition of functions, a genuinely non-commutative operation. Our "fundamental theorem" then yields an explicit factorization into primes such that uniqueness up to commutation is replaced by a non-commutative version of uniqueness.

Before making this precise, let us explain our concept of arithmetic which appears to be another instance of *quantum B-algebras*, a class of partially ordered algebras with two binary operations ($\to$ and $\rightsquigarrow$, featuring non-commutative implication) which can be characterized by their embeddability into quantales [32]. For the relevance and application of quantum B-algebras and their unifying rôle in algebraic logic, we refer to [32,33].

An ideal I of an arbitrary ring S is said to be *invertible* if it belongs to the Picard group $\mathrm{Pic}(S)$, that is, I is a progenerator defining a self-equivalence of the category of left S-modules. The (S,S)-subbimodules of the ring $\widetilde{S} := \varinjlim \mathrm{Hom}_S(I,S)$ (where I runs through the set of invertible ideals) form a quantale $Q(S)$ with compact unit element S. To any quantale Q with compact unit element u we associate its *arithmetic* $\mathscr{A}(Q)$, consisting of the elements $a \in A$ for which there is an invertible element $\alpha \in Q^\times$ with $\alpha \leqslant a \leqslant \alpha^{-1}$. In particular, every ring S has an *arithmetic* $\mathscr{A}(S) := \mathscr{A}(Q(S))$. For an R-order Λ, the arithmetic $\mathscr{A}(\Lambda)$ consists of the fractional ideals. In this case, the two operations $I \to J$ and $I \rightsquigarrow J$ of the arithmetic $\mathscr{A}(\Lambda)$ coincide with the left and right ideal quotient $(J : I)_\ell$ and $(J : I)_r$, respectively. An arithmetic A is called *hereditary* if the equations