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Graded polynomial identities for matrices with the transpose involution

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ABSTRACT

Let G be a group of order k. We consider the algebra $M_k(\mathbb{C})$ of k by k matrices over the complex numbers and view it as a crossed product with respect to G by embedding G in the symmetric group S_k via the regular representation and embedding S_k in $M_k(\mathbb{C})$ in the usual way. This induces a natural G-grading on $M_k(\mathbb{C})$ which we call a crossed-product grading. We study the graded $*$-identities for $M_k(\mathbb{C})$ equipped with such a crossed-product grading and the transpose involution. To each multilinear monomial in the free graded algebra with involution we associate a directed labeled graph. Use of these graphs allows us to produce a set of generators for the $(T, *)$-ideal of identities. It also leads to new proofs of the results of Kostant and Rowen on the standard identities satisfied by skew matrices. Finally we determine an asymptotic formula for the $*$-graded codimension of $M_k(\mathbb{C})$.

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1. Introduction

In this paper we consider graded $*$-identities on the algebra $M_k(\mathbb{C})$ of $k \times k$ matrices over the complex numbers. A $*$-identity (ignoring the grading) is a polynomial over $\mathbb{Q}$ in variables x_i and x_i^* which vanishes on every substitution of matrices in $M_k(\mathbb{C})$, with the condition that if the matrix A is substituted for x_i, then A^* (the transpose of A) is substituted for x_i^*. Such identities have been studied by several authors. If G is a group we say a $\mathbb{C}$-algebra B is G-graded if there is, for each $g \in G$, a subspace B_g of B (possibly zero) such that $B = \oplus_{g\in G} B_g$ and for all $g, h \in G$, $B_g B_h \subseteq B_{gh}$. Given a G-grading on $M_k(\mathbb{C})$ we may consider G-graded $*$-identities, that is, polynomials in weighted variables $x_{i,g}$ and $x_{i,g}^*$ that vanish under all homogeneous substitutions. In other words we substitute elements from the homogeneous component $M_k(\mathbb{C})_g$ for a variable $x_{i,g}$ and if we substitute A for $x_{i,g}$ we must substitute A^* for $x_{i,g}^*$. We are interested in a particular grading, the crossed-product grading, on $M_k(\mathbb{C})$ given by a group G of order k: To produce this grading we begin by imbedding G, via the regular representation, in the group P_k of $k \times k$ permutation matrices. We then set $M_k(\mathbb{C})_g = D_k P_g$ where D_k denotes the set of diagonal matrices and P_g is the permutation matrix corresponding to $g \in G$. For more information concerning crossed product gradings and, in particular, their relation to elementary gradings, we refer the reader to the Introduction of [2]. Our main objects of study are the graded $*$-identities on $M_k(\mathbb{C})$ endowed with the G-crossed product grading.

Our approach is to use graph theory in a way analogous to our work on G-graded identities on $M_k(\mathbb{C})$, where the G-grading is the crossed-product grading described above. We start with the free algebra $\mathbb{Q}\{x_{i,g}, x_{i,g}^* | i \geq 1, g \in G\}$, which is a G-graded algebra and admits an obvious involution, also denoted by $*$. In this free algebra we consider $I(G,*)$, the ideal of G-graded $*$-identities for $M_k(\mathbb{C})$. This ideal is a $(T,*)$-ideal, which means that it is invariant under graded endomorphisms and under the involution $*$. As in the classical case one can show that $I(G,*)$ is generated as a $(T,*)$-ideal by the set of *strongly multilinear* identities, where a strongly multilinear polynomial is a polynomial of the following form:

$$\sum_{\pi \in S_n} a_\pi x_{\pi(1),\sigma_{\pi(1)}}^{\epsilon_{\pi,1}} x_{\pi(2),\sigma_{\pi(2)}}^{\epsilon_{\pi,2}} \cdots x_{\pi(n),\sigma_{\pi(n)}}^{\epsilon_{\pi,n}}$$

where each a_π is a rational number and for each pair (π, j), $\epsilon_{\pi,j}$ is either nothing or $*$. The adjective "strongly" is to indicate that these monomials are not just multilinear in the variables $x_{i,g}$ and $x_{i,g}^*$ but also in the numerical subscripts, so that we do not allow $x_{i,g}^{\epsilon_i}$ and $x_{i,h}^{\gamma_i}$ to appear in the same monomial unless $g = h$ and $\epsilon_i = \gamma_i$. As in the case of graded identities without involution we associate a finite directed graph to each strongly multilinear monomial. It turns out that two strongly multilinear monomials have the same graph if and only if the difference of the monomials is a graded $*$-identity.

We give three main applications. The first is a determination of a very simple set of identities that generate the $(T,*)$-ideal of graded $*$-identities for $M_k(\mathbb{C})$ (see Theorem 8).