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Post-Lie algebra structures on pairs of Lie algebras

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We study post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, which describe simply transitive nil-affine actions of Lie groups. We prove existence results for such structures depending on the interplay of the algebraic structures of $\mathfrak{g}$ and $\mathfrak{n}$. We consider the classes of simple, semisimple, reductive, perfect, solvable, nilpotent, abelian and unimodular Lie algebras. Furthermore we consider commutative post-Lie algebra structures on perfect Lie algebras. Using Lie algebra cohomology we can classify such structures in several cases. We also study commutative structures on low-dimensional Lie algebras and on nilpotent Lie algebras.

1. Introduction

Post-Lie algebras and post-Lie algebra structures are an important generalization of left-symmetric algebras (also called pre-Lie algebras) and left-symmetric algebra struc-

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tures on Lie algebras, which arise in many areas of algebra and geometry [6], such as left-invariant affine structures on Lie groups, affine crystallographic groups, simply transitive affine actions on Lie groups, convex homogeneous cones, faithful linear representations of Lie algebras, operad theory and several other areas. A particular interesting problem with a long history concerns crystallographic groups and crystallographic structures on groups. Here a Euclidean crystallographic structure on a group Γ is a representation $\rho\colon \Gamma \to \mathrm{Isom}(\mathbb{R}^n)$ letting Γ act properly discontinuously and cocompactly on $\mathbb{R}^n$. By the Bieberbach theorems, the groups admitting a Euclidean crystallographic structure are precisely the finitely generated virtually abelian groups. The torsion-free crystallographic groups, called Bieberbach groups, are exactly the fundamental groups of compact flat Riemannian manifolds. John Milnor addressed the question how this result can be generalized to *affine crystallographic structures* for Γ, i.e., for representations $\rho\colon \Gamma \to \mathrm{Aff}(\mathbb{R}^n)$ letting Γ act properly discontinuously and cocompactly on $\mathbb{R}^n$. More precisely he asked whether or not every virtually polycyclic group admits an affine crystallographic structure. Although there was a lot of evidence for Milnor's question having a positive answer, counterexamples were found [3,4], in particular in terms of left-symmetric structures on Lie algebras. For the history and references see [6]. Another natural question then was whether one could find a reasonable class of generalized affine structures on Γ such that Milnor's question would have a positive answer. This was indeed possible and led to the notion of a so-called *nil-affine crystallographic structure* for Γ, i.e., a representation $\rho\colon \Gamma \to \mathrm{Aff}(N)$, letting Γ act properly discontinuously and cocompactly on a simply connected, connected nilpotent Lie group N. Here $\mathrm{Aff}(N) = N \rtimes \mathrm{Aut}(N)$ denotes the affine group of N. It was shown in [13] and [1] that every virtually polycyclic group admits a nil-affine crystallographic structure. The natural generalization of left-symmetric structures, which correspond to the case $N = \mathbb{R}^n$, are exactly the post-Lie algebra structures on pairs of Lie algebras. This is the reason that these structures are so fundamental for nil-affine crystallographic structures. In the nilpotent case, these structures can also be formulated by nil-affine actions of a nilpotent Lie group G on another nilpotent Lie group N, and such Lie group actions can be translated to the level of Lie algebras leading directly to post-Lie algebra structures [7,10].

As in the case of left-symmetric structures, the existence question of post-Lie algebra structures is very important (and very hard in general). We already studied an intermediate case, the so-called LR-structures on Lie algebras in [8,9], and have obtained first results for the general case in [10,12]. In this paper we are able to prove more general results. One result concerns perfect Lie algebras $\mathfrak{g}$, i.e., satisfying $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. It has been proved in [21] that a perfect Lie algebra over a field of characteristic zero does not admit a left-symmetric structure. We extend this result and show that if $\mathfrak{g}$ is perfect and $\mathfrak{n}$ is solvable and non-nilpotent, then there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. We will also prove that if $(\mathfrak{g}, \mathfrak{n})$ is a pair of Lie algebras, where $\mathfrak{g}$ is simple, and $\mathfrak{n}$ is not isomorphic to $\mathfrak{g}$, then there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. This shows that the assumption that $\mathfrak{g}$ is simple is very strong. Already the case that $\mathfrak{n}$ is semisimple has