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Valuations on rational function fields that are invariant under permutation of the variables ☆

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We study and characterize the class of valuations on rational functions fields that are invariant under permutation of the variables and can be extended to valuations with the same property whenever a finite number of new variables is adjoined. The Gauß valuation is in this class, which constitutes a natural generalization of the concept of Gauß valuation. Further, we apply our characterization to show that the most common ad hoc generalization of the Gauß valuation is also in this class.

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1. Introduction

In this paper, we will work with (Krull) valuations v and write them in the classical additive way, that is, the value group of v on a field K, denoted by vK, is an additively

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written ordered abelian group, and the ultrametric triangle law reads as $v(a+b) \geq \min\{va, vb\}$. We denote by Kv the residue field of v on K, by va the value of an element $a \in K$, and by av its residue.

Take any field K and a rational function field $K(X_1, \ldots, X_m)$. In [6], the possible extensions of a valuation on K to $K(X_1, \ldots, X_m)$ have been studied. While a quite good description of all possible value groups and residue fields of the extensions has been achieved, a detailed description and characterization of the extensions themselves seems rather out of reach. In this situation, restricting the problem to a subset of extensions with additional properties may well be of help.

In situations where at least one extension of a nontrivial valuation of K has to be constructed on $K(X_1, \ldots, X_m)$, the **Gauß valuation** is often the canonical choice. Once we have a valuation on a polynomial ring, it extends in a unique way to the quotient field. On $K[X_1, \ldots, X_m]$, the Gauß valuation is defined as follows. Given $f \in K[X_1, \ldots, X_m]$, write

$$f = \sum_{\underline{i}} d_{\underline{i}} X_1^{i_1} \cdot \ldots \cdot X_m^{i_m} \tag{1}$$

where the sum runs over multi-indices $\underline{i} = (i_1, \ldots, i_m)$ and each $d_{\underline{i}}$ is an element of K. Then define

$$vf := \min_{\underline{i}} vd_{\underline{i}} \, . \tag{2}$$

This indeed defines a valuation on $K[X_1, \ldots, X_m]$; see Corollary 2.2 below.

We have occasionally witnessed ad hoc attempts to generalize the concept of Gauß valuation, but these attempts did not consider its particular properties in any systematic way. So let us take a closer look at some of them.

First, we notice that definition (2) is invariant under permutation of the variables. Second, the same definition can be used to extend the valuation to $K[X_1, \ldots, X_n]$ for each $n > m$, preserving the property of being invariant. Third, it is well known (and follows from Lemma 2.4 below) that for the Gauß valuation v on $K[X_1, \ldots, X_n]$, the residues $X_1 v, \ldots, X_m v$ are algebraically independent over the residue field Kv. This implies that v is an Abhyankar valuation on the function field $K(X_1, \ldots, X_n)|K$; the definition of this notion will be given after Theorem 1.3.

The property that the residues $X_1 v, \ldots, X_m v$ are algebraically independent over Kv singles out the Gauß valuation. In order to obtain a generalization of the notion of Gauß valuation, we have to drop this property. Our goal therefore is to characterize the valuations that have the first two properties. Thereafter we will clarify their relation to the property of being Abhyankar valuations.

For each permutation $\pi \in S_m$ we denote by τ_π the automorphism of $K(X_1, \ldots, X_m)$ over K induced by the corresponding permutation

$$(X_1, \ldots, X_m) \mapsto (X_{\pi(1)}, \ldots, X_{\pi(m)}) \, .$$

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