Journal of Algebra 464 (2016) 348–356

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Units of p-power order in principal p-blocks of p-constrained groups

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ARTICLE INFO

Article history:
Received 1 October 2008
Available online 20 December 2012
Communicated by Leonard L. Scott, Jr.

Keywords:
Principal block
Torsion unit
Permutation lattice
p-constrained group

ABSTRACT

Let G be a finite group having a normal p-subgroup N that contains its centralizer $C_G(N)$, and let R be a p-adic ring. It is shown that any finite p-group of units of augmentation one in RG which normalizes N is conjugate to a subgroup of G by a unit of RG, and if it centralizes N it is even contained in N.

1. Introduction

This paper grew out of an attempt to understand more fully part of a theorem due to Roggenkamp and Scott (see [11, Theorem 6], [17,18], [12, Theorem 19]) about conjugacy of certain finite p-subgroups in the group of units of a p-adic group ring. The theorem in question, by now called the F*-Theorem, is stated below together with references where a detailed account on its proof can be found (with the result for $p = 2$ from Section 2 of the present paper being relevant[2]).

Some of the interesting aspects of the group of units of a group ring SG of a finite group G concern its finite subgroups, in particular when the coefficient ring S is a G-adapted ring, i.e., an

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1 Research supported by the Deutsche Forschungsgemeinschaft.

2 The communicating editor notes, with permission of the author, that Section 2 of this paper corrects an error in the $p = 2$ case of the proof of the F*-Theorem described in a 1987 manuscript of Roggenkamp and Scott. (This is Ref. [23] in [4]. The manuscript was not widely circulated, though there were published summaries [11,12,15].) The authors of [4] note that the $p = 2$ difficulty is avoidable in the original publicly announced version [17] of the F*-Theorem, which starts from $\mathbb{Z}G$ or a semilocalization over the prime divisors of the group order, rather than a group algebra over a p-adic ring. While this earlier version is adequate for most needs regarding the traditional group ring isomorphism problem, the stronger version, whose proof is completed in the present paper, along with new variations, is more important for isomorphism and unit group questions related to block theory and its applications. See, for instance, [18], and in Robinson's papers [9,10].

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.04.025

integral domain of characteristic zero in which no prime divisor of the order of G is invertible. Below, a few well known results in this case are listed. Note that it suffices to consider only the group of units $\mathrm{V}(SG)$ consisting of units of augmentation one. For $u \in \mathrm{V}(SG)$, we say that u is a trivial unit if $u \in G$, and the trace of u is its 1-coefficient (with respect to the basis G). Let S be a G-adapted ring. Then (see [16,8] or [19]):

(a) a non-trivial unit of SG of finite order has trace zero;
(b) the order of a finite subgroup of $\mathrm{V}(SG)$ divides the order of G;
(c) a central unit of finite order is a trivial unit.

Limiting attention only to finite p-subgroups in the group of units, one might ask whether comparable results hold with S replaced by the ring $\mathbb{Z}_p$ of p-adic integers. However, (a) and (b) do not carry over, even not if $\mathbb{Z}_pG$ consists of a single block only (cf. [15, Section XIV]). Imposing additional conditions one might also ask how certain finite p-subgroups are embedded in $\mathrm{V}(\mathbb{Z}_pG)$. If, for example, attention is directed to the principal block B of $\mathbb{Z}_pG$, a Sylow p-subgroup P of G is identified with its projection on B, and α is an augmented automorphism of B, then the question whether P is conjugate by a unit of B to its image $P\alpha$, is part of Scott's "defect group (conjugacy) question" (see [17, p. 267], [18]). For p-groups G, this question was answered in the affirmative by Roggenkamp and Scott [14].

Here, the following two theorems are proved. In both we assume that G has a normal p-subgroup N satisfying $\mathrm{C}_G(N) \leqslant N$. By definition, this means that G is p-constrained and $\mathrm{O}_{p'}(G) = 1$ (see [6, VII, 13.3]).

Let R be a p-adic ring, that is, the integral closure of the p-adic integers $\mathbb{Z}_p$ in a finite extension field of the p-adic field $\mathbb{Q}_p$. (Then R is a complete discrete valuation ring.) Note that by our assumption on G, the group ring RG will consist of a single (principal) block only (see [6, VII, 13.5]).

Theorem A. *Suppose that G has a normal p-subgroup N that contains its centralizer $\mathrm{C}_G(N)$. Then any finite p-group in $\mathrm{V}(RG)$ which normalizes N is conjugate to a subgroup of G by a unit of RG.*

Theorem B. *Suppose that G has a normal p-subgroup N that contains its centralizer $\mathrm{C}_G(N)$. Then any finite p-group in $\mathrm{V}(RG)$ which centralizes N is contained in N.*

Theorem B is both a corollary of Theorem A and used in its proof. To deduce it immediately from Theorem A, let C be a finite p-group in $\mathrm{V}(RG)$ which centralizes N. Then by Theorem A, $\langle N, C\rangle^u \leqslant G$ for some unit u of RG. Consequently, it follows that $N^u = N$ since N^u maps to 1 under the natural map $RG \to RG/N$. So $[N, C^u] = [N^u, C^u] = 1$ and $C^u \leqslant N = N^u$ by assumption on N, that is, $C \leqslant N$.

The proofs are somewhat complicated by the fact that we do not know in advance that RG is free for the "multiplication action" of the finite p-group under consideration (taking this for granted, Theorem A should be part of the F*-Theorem). Section 2 contains some preparatory results needed for the handling of the case $p = 2$. Theorems A and B are proved in Section 3. The bimodule arguments used there are inspired by [11, p. 231]. The proof depends heavily on the strong results of Weiss on p-permutation lattices (see [21,22,13]). The following theorem will be used.

Theorem *(Weiss). Let M be an R-representation of a finite p-group H. Suppose that N is a normal subgroup of H so that*

(a) *the restriction M_N of M to N is a free RN-module*;
(b) *the fixed point module M^N is a permutation lattice for G/N over R.*

Then M is a permutation lattice for G over R.

That this theorem can be applied rests upon the "Ward–Coleman Lemma." Coleman's contribution [1] is well known, but the first version of the lemma appears in an article of Ward [20] as