Journal of Algebra 463 (2016) 1–9

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Deformations of canonical triple covers

Francisco Javier Gallego [a], Miguel González [b,*], Bangere P. Purnaprajna [c]

[a] Departamento de Álgebra, Universidad Complutense de Madrid and Instituto de Matemática Interdisciplinar, Madrid, Spain
[b] Departamento de Álgebra, Universidad Complutense de Madrid, Madrid, Spain
[c] Department of Mathematics, University of Kansas, Lawrence, KS, USA

ARTICLE INFO

Article history:
Received 22 February 2016
Available online 30 June 2016
Communicated by Steven Dale Cutkosky

Dedicated to Lawrence Ein in his 60th birthday

Keywords:
Algebraic geometry
Projective varieties of general type

ABSTRACT

In this paper, we show that if X is a smooth variety of general type of dimension $m \geq 3$ for which the canonical map induces a triple cover onto Y, where Y is a projective bundle over $\mathbf{P}^1$ or onto a projective space or onto a quadric hypersurface, embedded by a complete linear series (except $\mathbf{Q}_3$ embedded in $\mathbf{P}^4$), then the general deformation of the canonical morphism of X is again canonical and induces a triple cover. The extremal case when Y is embedded as a variety of minimal degree is of interest, due to its appearance in numerous situations. For instance, by looking at threefolds Y of minimal degree we find components of the moduli of threefolds X of general type with $K_X^3 = 3p_g - 9, K_X^3 \neq 6$, whose general members correspond to canonical triple covers. Our results are especially interesting as well because they have no lower dimensional analogues.

* Corresponding author.
E-mail addresses: gallego@mat.ucm.es (F.J. Gallego), mgonza@mat.ucm.es (M. González), purna@math.ku.edu (B.P. Purnaprajna).

http://dx.doi.org/10.1016/j.jalgebra.2016.06.015
0021-8693/

Introduction

In this article we prove new results on deformations of canonical morphisms of degree 3 of a variety of general type of arbitrary dimension when the image Y of the morphism is a rational variety such as the projective space, a quadric hypersurface or a $\mathbf{P}^n$ bundle over a projective line. We give special emphasis on those Y which are varieties of minimal degree. This is a new case that does not occur in either curves or surfaces. Note that the degree of the canonical morphism for curves is bounded by 2. It was shown in [6] that there are no odd degree canonical covers of smooth surfaces of minimal degree other than linear $\mathbf{P}^2$. The geometric genus of a canonical triple cover of a singular surface of minimal degree is bounded by 5. So the geometry of the higher dimensional covers has no analogue in lower dimensions. In this article we handle more general canonical triple covers since Y need not be of minimal degree. It is a well known fact that low degree covers of varieties of minimal degree have a ubiquitous presence in the geometry of varieties of general type, especially in the case of algebraic surfaces and threefolds such as Calabi–Yau. They appear in the geometry of higher dimensional varieties of general type as well, as [2] and [8] and the very recent work of [1] show. These deal mostly with the case of double covers. The work [3] deals with structure theorem for triple covers.

The result of Castelnuovo for an algebraic surface of general type says that if the linear system $|K_X|$ is birational, then $K_X^2 \geq 3p_g - 7$. There is no known general result for a threefold of general type in these directions. Triple canonical covers of rational varieties satisfy the inequality $K_X^3 \geq 3p_g - 9$. There is a strong correlation in the numerology between surfaces and threefolds. It was shown in a very recent work in [1] that the geometry of Horikawa threefolds, which satisfy $K_X^3 = 2p_g - 6$, has striking similarities to Horikawa surfaces, that is, those surfaces which are on the Noether line $K_X^2 = 2p_g - 4$. This analogy is important for understanding the geometry of the moduli spaces of higher dimensional varieties. The numerology suggests that, for a threefold, the analogue of Castelnuovo's bound could be $K_X^3 \geq 3p_g - 9$. That is, for a threefold X, if the linear system $|K_X|$ induces a birational map, then K_X^3 would be greater than or equal to $3p_g - 9$. On the other hand, the equality $K_X^3 = 3p_g - 9$ (which holds for canonical triple covers X of threefolds Y of minimal degree) is sufficiently high for a threefold, so $|K_X|$ can be potentially (or conjecturally) birational in many cases. However, in this article we show that the deformations of canonical triple covers of threefolds Y of minimal degree, except if Y is a quadric threefold in $\mathbf{P}^4$, are not birational; instead, they are again morphisms of degree 3. Thus we find components of the moduli of threefolds X of general type with $K_X^3 = 3p_g - 9, K_X^3 \neq 6$, whose general members correspond to canonical triple covers (see Corollary 1.11). This is the situation in the extremal case when X is a canonical triple cover of a threefold Y embedded as a variety of minimal degree. If X is a canonical triple cover of a rational threefold Y not of minimal degree, then K_X^3 can be much higher than $3p_g - 9$. Even then, the results in this article show that the deformations of canonical triple covers of either projective bundles over $\mathbf{P}^1$ or projective spaces or quadric hypersurfaces, embedded by a complete linear series (with the exception of $\mathbf{Q}_3$