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Deformations of canonical double covers

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A B S T R A C T

In this paper we show that if X is a smooth variety of general type of dimension $m \geq 2$ for which its canonical map induces a double cover onto Y, where Y is the projective space, a smooth quadric hypersurface or a smooth projective bundle over $\mathbf{P}^1$, embedded by a complete linear series, then the general deformation of the canonical morphism of X is again canonical and induces a double cover. The second part of the article proves the non-existence of canonical double structures on the rational varieties above mentioned. Our results have consequences for the moduli of varieties of general type of arbitrary dimension, since they show that infinitely many moduli spaces of higher dimensional varieties of general type have an entire "hyperelliptic" component. This is in sharp contrast with the case of curves or surfaces of lower Kodaira dimension.

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Introduction

Canonical covers of rational varieties such as the projective space, projective bundles over a curve and so on have a ubiquitous presence in the geometry of varieties of general type. They occur in various situations and compellingly in the extremal cases, for example, in the case of the surfaces on the Noether line $K_X^2 = 2p_g - 4$. This was studied and classified by Horikawa in [9]. The higher dimensional analogues of these results have been proved in [3,11] and more recently in [1]. In all these cases it turns out that the varieties of general type on the Noether line are canonical double covers of varieties of minimal degree, that is, canonical double covers of certain rational varieties.

In this article, we study the deformations of the canonical morphism of degree 2 of a variety of general type of arbitrary dimension, when the image Y of the morphism is a rational variety that is either $\mathbf{P}^m$, a smooth projective bundle over a rational curve or a smooth hyperquadric, that is, we study the canonical double covers of these rational varieties. As mentioned above, they have a ubiquitous presence in the geometry of varieties of general type. We prove that the general deformation of the canonical morphism of degree 2 is again a canonical morphism of degree 2. Some of the cases regarding deformation results have an overlap with results in [3], but the methods and proofs are qualitatively different.

Some things are known about the components of the moduli space of surfaces of general type, although still there is a lot to be understood. However, the knowledge of the moduli of varieties of general type of higher dimension is largely in the nascent stage. The results of this article have implications to the moduli of varieties of general type. In [5] and [6], we constructed infinitely many moduli spaces of surfaces of general type with “hyperelliptic” components. There were moduli spaces with two kinds of components: one hyperelliptic component and another component whose general points corresponded to surfaces with degree 1 canonical morphism and with a special locus parameterizing surfaces whose canonical morphism was of degree 2. The results of this article do show (see Corollary 1.8) the existence of a hyperelliptic component in infinitely many moduli spaces of varieties of general type of arbitrary dimension. In [6] it was shown that double canonical covers of rational surfaces could have invariants with $K_X^2 > 3p_g - 7$. This is the Castelnuovo bound and any $|K_X|$ that is birational has to satisfy this inequality. In higher dimensions there is no such bound to the best of our knowledge, but it can be shown that, in threefolds for instance, double canonical covers can have $K_X^3 > 3p_g - 7$. There is indeed a correlation between numerology in various dimensions. It has been shown in a very recent work of Chen, Gallego and Purnaprajna [1] that the “Noether faces” for higher dimensions are $K_X^m = 2(p_g - m)$, and that this determines the precise analogues in higher dimensions of Horikawa’s results for surfaces. So, conjecturally at least, one should expect similar results for birationality of $|K_X|$ in higher dimensions, as Castelnuovo’s result for surfaces.

Higher multiplicity structures on smooth varieties occur very naturally in algebraic geometry, especially as degenerations of smooth varieties. They are also of significant