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Idempotent plethories

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ABSTRACT

Let k be a commutative ring with identity. A k-*plethory* is a commutative k-algebra P together with a comonad structure $\mathbb{W}_P$, called the P-*Witt ring* functor, on the covariant functor that it represents. We say that a k-plethory P is *idempotent* if the comonad $\mathbb{W}_P$ is idempotent, or equivalently if the map from the trivial k-plethory $k[e]$ to P is a k-plethory epimorphism. We prove several results on idempotent plethories. We also study the k-plethories contained in $K[e]$, where K is the total quotient ring of k, which are necessarily idempotent and contained in $\mathrm{Int}(k) = \{f \in K[e] : f(k) \subseteq k\}$. For example, for any ring l between k and K we find necessary and sufficient conditions—all of which hold if k is a integral domain of Krull type—so that the ring $\mathrm{Int}_l(k) = \mathrm{Int}(k) \cap l[e]$ has the structure, necessarily unique and idempotent, of a k-plethory with unit given by the inclusion $k[e] \longrightarrow \mathrm{Int}_l(k)$. Our results, when applied to the binomial plethory $\mathrm{Int}(\mathbb{Z})$, specialize to known results on binomial rings.

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1. Introduction

In this paper all rings and algebras, unless otherwise stated, are assumed commutative with identity. We denote the category of sets by Sets and the category of abelian groups by Ab. For any ring k we let k-Mod and k-Alg denote the category of k-modules and the category of k-algebras, respectively, and for any k-module M we denote the n-th tensor power of M over k by $M^{\otimes_k n}$, or $M^{\otimes n}$ if the ring k is understood.

Let k be a ring. A *k-plethory* is a k-algebra P together with a comonad structure $\mathbb{W}_P$, called the *P-Witt ring* functor, on the covariant functor $\mathrm{Hom}_{k\text{-}\mathsf{Alg}}(P, -)$ that it represents [5]. A k-plethory is also known as a *k-k-biring monoid (or monad object)*, a *k-k-biring triple*, and a *Tall–Wraith monoid (or monad object) in k-Alg* [3,40]. Trivially, the polynomial ring $k[X]$ has the structure of a k-plethory, denoted $k[e]$ and called the *trivial* k-plethory, which is an initial object in the category of k-plethories.

Motivated by our previous efforts [23] to use the theory of plethories to generalize our results in [20] on binomial rings, we say that a k-plethory P is *idempotent* if the comonad $\mathbb{W}_P$ is idempotent, in the sense of [2], [4, Definition 4.1.1], [18,33]; that is, P is idempotent if the natural transformation $\mathbb{W}_P \longrightarrow \mathbb{W}_P \circ \mathbb{W}_P$ is an isomorphism, or, equivalently, if the composition map $P \odot P \longrightarrow P$ is an isomorphism. The idempotent k-plethories are the plethystic analogue of the *k-epimorphs*, which are the k-algebras A such that the map $k \longrightarrow A$ is an epimorphism of rings, or equivalently such that the multiplication map $A \otimes_k A \longrightarrow A$ is an isomorphism [38, Theorem 1]. (The $\mathbb{Z}$-epimorphs were classified in [8] and again in [9], and the classification was later generalized in [19] to Dedekind domains.) Not surprisingly, an analogous equivalence holds for plethories: a k-plethory P is idempotent if and only if the map $k[e] \longrightarrow P$ from the trivial k-plethory to P is an epimorphism of k-plethories.

This paper represents a first step towards a classification of the idempotent k-plethories, or more generally the k-plethory epimorphisms. This problem is embedded in two larger problems: first, to generalize, when possible, results in commutative algebra and algebraic geometry to plethystic algebra, and, second, to classify all k-plethories, which recently has been solved for fields k of characteristic zero [13]—all such plethories are linear—and which could be within reach for $k = \mathbb{Z}$. Among our results are several equivalent characterizations of the idempotent plethories, namely, Theorems 2.9, 4.3, 6.4, and 6.7 and Propositions 5.2 and 6.6. In Section 2 we provide an overview of the paper, along with motivation for the theory from the standpoint of binomial rings and integer-valued polynomial rings, and in Section 3 we summarize the relevant definitions and theorems from the theory of plethories as presented in [5] by Borger and Wieland. Sections 4 and 5 focus on general results that have analogues for the k-epimorphs, and Section 6 is concerned with questions of existence and uniqueness of idempotent plethory structures. Sections 7 and 8 are devoted to the study of k-plethories contained in $K[e]$, where K is the total quotient ring of k, which are all necessarily idempotent and contained in $\mathrm{Int}(k) = \{f \in K[e] : f(k) \subseteq k\}$. There we provide, for example, some exotic examples of k-plethories for any Krull domain k, including not only $\mathrm{Int}(k)$ but also the