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Solutions of the Yang–Baxter equation associated with a left brace

David Bachiller [a,*], Ferran Cedó [a], Eric Jespers [b]

[a] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*
[b] *Department of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium*

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Given a left brace B, a method is given to construct explicitly all the non-degenerate involutive set-theoretic solutions (X, r) of the Yang–Baxter equation such that the associated permutation group $\mathcal{G}(X, r)$ is isomorphic, as a left brace, to B. This method depends entirely on the brace structure of B.

1. Introduction

The quantum Yang–Baxter equation is an important equation coming from theoretical physics, first appearing in the works of Yang [32] and Baxter [4]. Recall that a solution of the quantum Yang–Baxter equation is a linear map $R : V \otimes V \longrightarrow V \otimes V$, where V is a vector space, such that

* Corresponding author.
E-mail addresses: dbachiller@mat.uab.cat (D. Bachiller), cedo@mat.uab.cat (F. Cedó), efjesper@vub.ac.be (E. Jespers).

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$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where R_{ij} denotes the map $V \otimes V \otimes V \longrightarrow V \otimes V \otimes V$ acting as R on the (i,j) tensor factor and as the identity on the remaining factor. A central open problem is to construct new families of solutions of this equation. It is this problem which initially motivated the definition of quantum groups, and one of the reasons of the recent interest in Hopf algebras, see [24].

Let $\tau\colon V \otimes V \longrightarrow V \otimes V$ be the linear map such that $\tau(u \otimes v) = v \otimes u$ for all $u, v \in V$. Then it is easy to check that $R\colon V \otimes V \longrightarrow V \otimes V$ is a solution of the quantum Yang–Baxter equation if and only if $\overline{R} = \tau \circ R$ satisfies

$$\overline{R}_{12}\overline{R}_{23}\overline{R}_{12} = \overline{R}_{23}\overline{R}_{12}\overline{R}_{23}.$$

In this case, one says that $\overline{R}$ is a solution of the Yang–Baxter equation.

Note that if X is a basis of the vector space V, then a map $r\colon X \times X \longrightarrow X \times X$, such that

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where r_{ij} denotes the map $X \times X \times X \longrightarrow X \times X \times X$ acting as r on the (i,j) components and as the identity on the remaining component, induces a solution of the Yang–Baxter equation. In this case, one says that (X,r) (or r) is a set-theoretic solution of the Yang–Baxter equation. Drinfeld, in [12], posed the question of finding these set-theoretic solutions.

A subclass of this type of solutions, the non-degenerate involutive solutions, has received a lot of attention in the last years [9–11,14,16–19,22,23,25–27]. This class of solutions is not only studied for the applications of the Yang–Baxter equation in physics, but also for its connection with other topics in mathematics of recent interest: semigroups of I-type and Bieberbach groups [19], bijective 1-cocycles [14], radical rings [27], triply factorized groups [30], construction of semisimple minimal triangular Hopf algebras [13], regular subgroups of the holomorf and Hopf–Galois extensions [8,15], and groups of central type [5,6].

Gateva-Ivanova and Van den Bergh [19], and Etingof, Schedler and Soloviev [14] introduced this subclass of solutions and associated with each such solution (X,r) of this type two groups which are fundamental for its study: the structure group $G(X,r)$, and the permutation group $\mathcal{G}(X,r)$. In order to study this class of solutions, Rump in [27] introduced a new algebraic structure, called a brace. Recall that a left brace is a set B with two operations, $+$ and $\cdot$, such that $(B,+)$ is an abelian group, $(B,\cdot)$ is a group and

$$x \cdot (y+z) + x = x \cdot y + x \cdot z, \tag{1}$$