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Fraïssé structures with universal automorphism groups

Isabel Müller

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ABSTRACT

We prove that the automorphism group of a Fraïssé structure M equipped with a notion of stationary independence is universal for the class of automorphism groups of substructures of M. Furthermore, we show that this applies to certain homogeneous n-gons.

1. Introduction

Certain homogeneous structures are universal with respect to the class of their substructures: The Rado graph is universal for the class of all countable graphs, the rationals as a dense linear order for the class of all countable linear orders and Urysohn's universal Polish space for the class of all Polish spaces. Jaligot asked whether a universal structure M transfers its universality onto its automorphism group, i.e. whether $\mathrm{Aut}(M)$ is universal for the class of automorphism groups of substructures of M (cf. [6]). Recently, Doucha showed that, for an uncountable structure M, the answer to Jaligot's question is

E-mail address: muelleri@uni-muenster.de.

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rarely positive [3]. Countable homogeneous structures on the contrary, most often have universal automorphism groups. In fact, the only known counterexample was pointed out by Piotr Kowalski and is given by the Fraïssé limit of finite fields in fixed characteristic p, which coincides with the algebraic closure of $\mathbb{F}_p$. Its automorphism group is $\hat{\mathbb{Z}}$, which is torsion free and hence does not embed any automorphism group of a finite field. It is still unknown if there is a relational countable counterexample.

We will prove that in the case where M is a Fraïssé structure admitting a certain stationary independence relation, the automorphism group $\mathrm{Aut}(M)$ will be universal for the class of automorphism groups of substructures of M.

Uspenskij [11], using a careful construction of Urysohn's universal Polish space given by Katětov [7], proved that its isometry group is universal for the class of all Polish groups, which corresponds to the class of isometry groups of Polish spaces [4]. The idea of Katětov thereby can be described as follows: Given a Polish space X, he constructed a new metric space $E_1(X)$ consisting of X together with all possible 1-point metric extensions, while assigning the smallest possible distance between new points. Under minor restrictions, the space obtained is again Polish, denoted by the first Katětov space of X. Iterating this, i.e. building one Katětov space over the other, he constructed a copy of Urysohn's space itself. Furthermore, all isometries of X extend in a unique way at every step of the construction, which yields the desired embedding of $\mathrm{Isom}(X)$ into $\mathrm{Isom}(\mathbb{U})$.

In [2] Bilge adapted this construction to Fraïssé limits of rational structures with free amalgamation by gluing extensions freely over the given space. Both Urysohn's spaces and Fraïssé classes with free amalgamation carry an independence relation as introduced by Tent and Ziegler [10]. In this paper, we will show that the mere presence of a stationary independence relation within a Fraïssé structure M allows us to mimic Katětov's construction of Urysohn's universal metric space, starting with any structure X embeddable in M. With the help of the given independence relation, we will glue "small" extensions of X independently and construct an analog of Katětov spaces in the non-metric setting, thereby ensuring that the automorphisms of X extend canonically to its Katětov spaces and that the extensions behave well under composition. In particular, we will give a positive answer to the question of Jaligot for the class of Fraïssé limits with stationary independence relation by proving the following result (Theorem 4.9):

Theorem. *Let M be a Fraïssé structure with stationary independence relation and $\mathcal{K}_\omega$ the class of all countable structures embeddable into M. Then for any $X \in \mathcal{K}_\omega$, there is an embedding $f : X \to M$ such that every automorphism of $f(X)$ extends to an automorphism of M in such a way that this extension yields a continuous embedding of* $\mathrm{Aut}(X)$ *into* $\mathrm{Aut}(M)$*. In particular, the automorphism group* $\mathrm{Aut}(M)$ *is universal for the class* $\mathrm{Aut}(\mathcal{K}_\omega) := \{\mathrm{Aut}(X) \mid X \in \mathcal{K}_\omega\}$*, i.e. every group in* $\mathrm{Aut}(\mathcal{K}_\omega)$ *can be continuously embedded as a subgroup into* $\mathrm{Aut}(M)$*.*

Note, that every automorphism group of a countable first order structure M can be considered as a Polish group if we equip it with the topology of pointwise convergence.