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Relative BGG sequences: I. Algebra ☆

Andreas Čap [a,*], Vladimir Souček [b]

[a] Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
[b] Mathematical Institute, Charles University, Sokolovská 83, Praha, Czech Republic

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We develop a relative version of Kostant's harmonic theory and use this to prove a relative version of Kostant's theorem on Lie algebra (co)homology. These are associated to two nested parabolic subalgebras in a semisimple Lie algebra. We show how relative homology groups can be used to realize representations with lowest weight in one (regular or singular) affine Weyl orbit. In the regular case, we show how all the weights in the orbit can be realized as relative homology groups (with different coefficients). These results are motivated by applications to differential geometry and the construction of invariant differential operators.

1. Introduction

This article is the first in a series of two. The main aim of the series is to develop a relative version of the machinery of Bernstein–Gelfand–Gelfand sequences (or BGG

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* Corresponding author.
E-mail addresses: Andreas.Cap@univie.ac.at (A. Čap), soucek@karlin.mff.cuni.cz (V. Souček).

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sequences) as introduced in [6] and [3] and to improve the original constructions at the same time, which is done in [7]. This is a construction for invariant differential operators associated to a class of geometric structures known as parabolic geometries. For each type of parabolic geometries, there is a homogeneous model, which is a generalized flag manifold, i.e. the quotient of a (real or complex) semisimple Lie group G by a parabolic subgroup P. The starting point for the construction of a BGG sequence is a finite-dimensional representation $\mathbb{V}$ of G. On the homogeneous model, the resulting sequence is a resolution of the locally constant sheaf $\mathbb{V}$ on G/P by differential operators acting on spaces of sections of homogeneous vector bundles induced by irreducible representations of P. The resulting resolution of $\mathbb{V}$ by principal series representations of G is dual (in a certain sense) to the resolution of $\mathbb{V}^*$ by generalized Verma-modules obtained in [11]. This generalizes the Bernstein–Gelfand–Gelfand resolution of $\mathbb{V}^*$ by Verma-modules from [1], which motivated the name of the construction.

The algebraic character of BGG sequences is also reflected by the tools needed for their construction, and this first part of the series is devoted to developing the necessary algebraic background for the relative version. In particular, we prove a relative version of Kostant's theorem on Lie algebra cohomology from [9], which should be of independent interest. The setup for Kostant's original theorem is a complex semisimple Lie algebra $\mathfrak{g}$, a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with (reductive) Levi-decomposition $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and a complex irreducible representation $\mathbb{V}$ of $\mathfrak{g}$. Then Kostant considered the standard complex $(C^*(\mathfrak{p}_+, \mathbb{V}), \partial)$ computing the Lie algebra cohomology of the nilpotent Lie algebra $\mathfrak{p}_+$ with coefficients in $\mathbb{V}$. The spaces in this complex are naturally representations of $\mathfrak{g}_0$ and the differentials are $\mathfrak{g}_0$-equivariant. Thus the cohomology groups $H^*(\mathfrak{p}_+, \mathbb{V})$ are representations of the reductive Lie algebra $\mathfrak{g}_0$ and Kostant's theorem describes these representations explicitly and algorithmically in terms of highest weights.

While higher Lie algebra cohomology groups seem to be difficult to interpret in general, Kostant's theorem has immediate algebraic applications, see [9]. Even the version for the Borel subalgebra (which in some respects is significantly simpler than the general result) very quickly implies the Weyl character formula, thus providing a completely algebraic proof for this formula. Moreover, together with the Peter–Weyl theorem, Kostant's theorem can be used to proof Bott's generalized version (see [2]) of the Borel–Weil theorem describing the sheaf cohomology of the sheaf of local holomorphic sections of a homogeneous vector bundle over a complex generalized flag manifold. Apart from the applications in the theory of parabolic geometries (see [5]), Kostant's theorem has also been applied in other areas recently. For example, in [10], Lie algebra cohomology as computed via Kostant's theorem is used as a replacement for Spencer cohomology in connection with exterior differential systems to prove rigidity results in algebraic geometry.

It is important to point out here that for the applications to parabolic geometries, not only Kostant's theorem itself is important. Also some of the tools introduced by Kostant in the proof play a central role there. These tools are also available for parabolic subalgebras in real semisimple Lie algebras, and the real versions are needed in the