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On compact exceptional objects in derived module categories

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ABSTRACT

Let A be a basic and connected finite dimensional algebra and $D^b(A)$ be the bounded derived category of finitely generated left A-modules. In this paper we consider lengths of tilting objects and indecomposable compact exceptional objects in $D^b(A)$, and prove a sufficient condition such that these lengths are bounded by the number of isomorphism classes of simple A-modules. Moreover, we show that algebras satisfying this criterion are bounded derived simple, and describe an algorithm to construct a family of algebras satisfying this condition.

1. Introduction

In algebraic representation theory, one of the most interesting and complicated problems is to classify algebras up to derived equivalence. Explicitly, given a finite dimensional

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(connected and basic) k-algebra A, we want to characterize or construct all (connected and basic) algebras whose bounded derived module categories are triangulated equivalent to the bounded derived category $D^b(A)$. This big project draws the attention of many people, and quite a few results have been obtained in two different approaches. In one direction, people have classified certain types of algebras with special properties; see [1,2,8,11–13,19,20]. In the other direction, several properties have been shown to be invariant under derived equivalence, such as number of isomorphism classes of simple modules and finiteness of global dimensions ([16]), finiteness of finitistic dimensions ([30]), finiteness of strong global dimensions ([18]), self-injective property ([3]), etc. However, there are much more questions unsolved. For instance, a finite dimensional local algebra is only derived equivalent to algebras Morita equivalent to itself (see [34]), and we will show that path algebras of Kronecker quivers have this property as well (see Example 5.1). But a complete list of basic algebras which are only derived equivalent to algebras Morita equivalent to themselves is not available yet.

According to a fundamental result of Rickard ([31,32]), an algebra Γ is derived equivalent to A if and only if there is a tilting object $T \in D^b(A)$ such that Γ is isomorphic to the opposite algebra of $\mathrm{End}_{D^b(A)}(T)$. Therefore, tilting objects, and more generally, compact exceptional objects are of particular importance, and hence are extensively studied. For instance, Angeleri Hügel, Koenig, and Liu (in [4–6]) use them to investigate recollements and stratifications of derived categories; and Al-Nofayee and Rickard point out in [3,33] that for a fixed algebra, there are at most countably many *basic* tilting objects T up to isomorphism and degree shift, where T is basic if its direct summands are pairwise nonisomorphic.

In this paper we mainly focus on *lengths* of objects in derived categories, which are defined as follows. For an arbitrary $P^\bullet \in K^-({}_A\mathcal{P})$, the homotopy category of right bounded complexes of finitely generated projective A-modules, let

$$a(P^\bullet) = \sup\{i \in \mathbb{Z} \mid P^i \neq 0\} - \inf\{i \in \mathbb{Z} \mid P^i \neq 0\},$$

called the *amplitude* of $P^\bullet$ ([9]). Since $K^-({}_A\mathcal{P})$ and the right bounded derived category $D^-(A)$ are equivalent as triangulated categories, for $X \in D^-(A)$, we define its length to be [2]

$$l(X) = \inf\{a(P^\bullet) + 1 \mid P^\bullet \in K^-({}_A\mathcal{P}) \text{ is quasi-isomorphic to } X\}.$$

Clearly, an object $X \in D^-(A)$ has finite length if and only if X is quasi-isomorphic to a certain $P^\bullet \in K^b({}_A\mathcal{P})$, or equivalently, X is compact.

Happel and Zacharia prove in [18] that lengths of all indecomposable objects in $D^b(A)$ are bounded if and only if A is *piecewise hereditary*; that is, $D^b(A)$ is equivalent to $D^b(\mathcal{H})$,

[2] Note that our definition of lengths is slightly different from that in [18]. The length defined here counts terms between the first nonzero term (if it exists) and the last nonzero term, whereas the length defined in [18] counts the number of differential maps between the first nonzero term (if it exists) and the last nonzero term. For a compact object, the difference of these two lengths is exactly 1.