Journal of Algebra 465 (2016) 41–61

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Finite p-groups with some isolated subgroups

Zvonimir Janko

Mathematical Institute, University of Heidelberg, 69120 Heidelberg, Germany

ARTICLE INFO

Article history:
Received 10 October 2015
Available online 22 July 2016
Communicated by E.I. Khukhro

MSC:
20D15

Keywords:
Finite p-groups
Isolated subgroups
Minimal nonabelian subgroups
Maximal abelian subgroups
Hamiltonian groups
Quasidihedral 2-groups
Hughes subgroup

ABSTRACT

We say that a subgroup H is isolated in a group G if for each $x \in G$ we have either $x \in H$ or $\langle x \rangle \cap H = \{1\}$. Here we shall determine certain classes of finite nonabelian p-groups which possess some isolated subgroups (Theorems 1, 2, 4, 6 and 8 to 16).

Let G be a finite p-group. We say (according to Y. Berkovich) that a subgroup H of G is isolated in G if for each $x \in G$ we have either $x \in H$ or $\langle x \rangle \cap H = \{1\}$. We shall determine certain classes of finite nonabelian p-groups which possess some isolated subgroups (Theorems 1, 2, 4, 6 and 8 to 16). Most of these classes have been predicted in Problems of Y. Berkovich stated in [3] (problem numbers: 3312, 3358, 3359, 3360, 3365). However, the concept of isolated subgroups appears much earlier (see for instance [5]).

All groups considered here will be finite p-groups and our notation is standard (see [1]). In particular, E_{p^k} will denote an elementary abelian group of order p^k, $\mathrm{S}(p^3)$ $(p > 2)$,

E-mail address: janko@mathi.uni-heidelberg.de.

http://dx.doi.org/10.1016/j.jalgebra.2016.06.032
0021-8693/

is a nonabelian group of order p^3 and exponent p, D_{2^n}, $n \geq 3$, is a dihedral group of order 2^n,

$$\mathrm{M}_{p^n} = \langle a, b \mid a^{p^{n-1}} = b^p = 1,\ [a, b] = a^{p^{n-2}} \rangle,$$

where $n \geq 3$, and in case $p = 2$, $n \geq 4$. Further, a quasidihedral group is a 2-group G with an abelian subgroup H of exponent > 2 and index 2 and there is an involution $i \in G - H$ which inverts each element in H. Finally, the Hughes subgroup $\mathrm{H}_p(G)$ of a p-group G is the subgroup of G generated by all elements of order $\neq p$.

Theorem 1. *Let G be a nonabelian p-group of exponent $> p$ all of whose maximal abelian subgroups of exponent $> p$ are isolated in G. Then G has an abelian maximal subgroup A of exponent $> p$ such that $A = \mathrm{H}_p(G)$ (Hughes subgroup).*

Proof. Let G be a nonabelian p-group of exponent $> p$ all of whose maximal abelian subgroups of exponent $> p$ are isolated in G. Then this condition is hereditary to all nonabelian subgroups of exponent $> p$. Let A be a maximal abelian subgroup of exponent $> p$ and let $A < B \leq G$ with $|B : A| = p$. Then all elements in $B - A$ are of order p and so $A = \mathrm{H}_p(B)$ and A is characteristic in B.

Suppose that $B < G$ and let $B < C \leq G$ with $|C : B| = p$ so that $A \trianglelefteq C$ and $|C/A| = p^2$. If $C/A \cong \mathrm{C}_{p^2}$, then all elements in $C - B$ are of order p^2 so that $\Omega_1(C) = B$. In that case consider an element $c \in C - B$ of order p^2 so that $B = A\langle c^p \rangle$ and $C = A\langle c \rangle$. Let $X < C$ be a maximal abelian subgroup in C containing $\langle c \rangle$ so that $AX = C$ and X is isolated in C. Let $X < Y \leq C$ with $|Y : X| = p$ so that all elements in $Y - X$ are of order p. But then C is generated by elements of order p in $(B - A) \cup (Y - X)$, contrary to $\Omega_1(C) = B$. It follows that we must have $C/A \cong \mathrm{E}_{p^2}$. In that case all elements in $C - A$ are of order p and so $\mathrm{H}_p(C) = A$. Also note that $C' \leq A$ and so C is metabelian. But this contradicts a result of Hogan–Kappe [4] stating that in a metabelian p-group the index of a nontrivial Hughes subgroup is at most p. It follows that we must have $B = G$ and so our theorem is proved. □

Theorem 2. *Let G be a nonabelian p-group of exponent $> p$ all of whose maximal abelian subgroups are isolated in their normalizers. Then $p > 2$ and G has an abelian maximal subgroup A of exponent $> p$ such that $A = \mathrm{H}_p(G)$.*

Conversely, all such p-groups satisfy the assumption of this theorem.

Proof. Let G be a nonabelian p-group of exponent $> p$ all of whose maximal abelian subgroups are isolated in their normalizers. Let A be a maximal normal abelian subgroup in G. Then $A < G$ and let $A < B \leq G$ with $|B : A| = p$, where all elements in $B - A$ are of order p.

Suppose $p = 2$. Then an involution $i \in B - A$ inverts each element in A. Since $\mathrm{C}_G(A) = A$, we have $\exp(A) > 2$ and $\mathrm{C}_A(i) = \Omega_1(A) < A$. Let X be a maximal abelian subgroup in G containing $\mathrm{C}_A(i) \times \langle i \rangle$, where $X \cap B = \mathrm{C}_A(i) \times \langle i \rangle$. Let $X < Y \leq AX$