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On threefolds isogenous to a product of curves ☆

Davide Frapporti, Christian Gleißner *

University of Bayreuth, Lehrstuhl Mathematik VIII, Universitätsstraße 30, D-95447 Bayreuth, Germany

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ABSTRACT

A *threefold isogenous to a product of curves* X is a quotient of a product of three compact Riemann surfaces of genus at least two by the free action of a finite group. In this paper we study these threefolds under the assumption that the group acts diagonally on the product. We show that the classification of these threefolds is a finite problem, present an algorithm to classify them for a fixed value of $\chi(\mathcal{O}_X)$ and explain a method to determine their Hodge numbers. Running an implementation of the algorithm we achieve the full classification of threefolds isogenous to a product of curves with $\chi(\mathcal{O}_X) = -1$, under the assumption that the group acts faithfully on each factor.

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* Corresponding author.
E-mail address: christian.gleissner@uni-bayreuth.de (C. Gleißner).

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Introduction

A complex algebraic variety X is *isogenous to a product of curves* if X is a quotient

$$X = (C_1 \times \ldots \times C_n)/G,$$

where the C_i's are compact Riemann surfaces of genus at least two and G is a finite group acting freely on $C_1 \times \ldots \times C_n$. This class of varieties of general type has been introduced by Catanese in [5]. Since then, a considerable amount of literature appeared, especially in the case of surfaces. In particular surfaces S isogenous to a product of curves with $\chi(\mathcal{O}_S) = 1$ (equivalently $p_g(S) = q(S)$) are completely classified, see [1,10,22,7,23,17,3].

A natural question is: "Is it possible to classify varieties X isogenous to a product for a fixed value of $\chi(\mathcal{O}_X)$ in higher dimensions?"

Let G^0 be the diagonal subgroup

$$G^0 := G \cap \big(\operatorname{Aut}(C_1) \times \ldots \times \operatorname{Aut}(C_n)\big).$$

There are two possibilities for the action of G on the product $C_1 \times \ldots \times C_n$:

- *Unmixed*: $G = G^0$, i.e. the group G acts diagonally.
- *Mixed*: $G \neq G^0$, i.e. there are elements in G permuting some factors of the product.

In the present article we concentrate on the unmixed case in dimension three, while in a forthcoming one we plan to investigate the mixed case.

Following the approach used by the above mentioned authors, we show that the classification problem can be translated into a problem of group theory.

Let $X := \big(C_1 \times C_2 \times C_3\big)/G$ be a threefold isogenous to a product of unmixed type. Restricting the G-action we obtain homomorphisms

$$\psi_i : G \to \operatorname{Aut}(C_i) \quad \text{with kernels} \quad K_i := \ker(\psi_i).$$

Note that in contrast to the surface case, it is not possible to assume that the kernels K_i are trivial.

To the threefold X we attach the *algebraic datum*

$$(G, K_1, K_2, K_3, V_1, V_2, V_3)$$

where V_i is a *generating vector* for the group G/K_i (see Definition 2.2).

Conversely, by Riemann's existence Theorem, a tuple consisting of a finite group G, three normal subgroups K_i and generating vectors V_i for G/K_i satisfying certain relations, determines a family of threefold isogenous to a product.

What turns the classification into a finite problem is the fact that the freeness assumption for the group action allows us to bound the group order in terms of $\chi(\mathcal{O}_X)$