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On low-degree representations of the symmetric group

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ABSTRACT

The aim of the present paper is to obtain a classification of all the irreducible modular representations of the symmetric group on n letters of dimension at most n^3, including dimension formulae. This is achieved by improving an idea, originally due to G. James, to get hands on dimension bounds, by building on the current knowledge about decomposition numbers of symmetric groups and their associated Iwahori–Hecke algebras, and by employing a mixture of theory and computation.

1. Introduction

It is a major open problem in representation theory of finite groups to understand the irreducible modular representations of the symmetric group $\mathcal{S}_n$ on n letters. Although

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considerable progress has been made in recent decades, not even their dimensions are in general known today. The situation becomes somewhat better if we restrict ourselves to the representations whose dimensions are bounded above by a polynomial in n: The aim of the present paper is to obtain an explicit classification of all the irreducible modular representations of $\mathcal{S}_n$ of degree at most n^3, together with explicit degree formulae, the final result being described in (6.3), and explicit lists in characteristic $p \leq 7$ being given in the subsequent sections from (6.4) on.

Our original motivation to investigate into this was a particular question asked a while ago by E. O'Brien, in view of the, at that time forthcoming, paper [10]. Accordingly, we are proud to be able to say that the present results are now used significantly in [10, Sect. 4]. From a broader perspective, the present paper lies well within the recent philosophy of collecting results on representations of low degree of finite simple groups and their close relatives, and aims at contributing to this programme; see for example [19,37], where both viewpoints of either fixing a degree bound, or, for the case of groups of Lie type, allowing for a polynomial degree bound in terms of the Lie rank, are pursued.

In order to achieve the goal specified above, it has turned out that quite a few additional pieces of information have to be collected or newly derived, in particular by improving an idea, originally due to G. James, to get hands on dimension bounds, by building on the current knowledge about decomposition numbers of symmetric groups and their associated Iwahori–Hecke algebras, and by applying a mixture of theoretical reasoning and computational techniques.

• The starting point of our considerations is an observation due to G. James [20], describing the growth behavior of irreducible modular representations of the symmetric group $\mathcal{S}_n$ on n letters, when n tends to infinity. More precisely:

Given a rational prime p, let $d^\mu := \dim_{\mathbb{F}_p}(D^\mu)$ be the dimension of the irreducible $\mathcal{S}_n$-module D^μ parameterized by the p-regular partition $\mu = [\mu_1, \mu_2, \ldots]$ of n. Fixing a non-negative integer m, and assuming that the largest part of μ equals $\mu_1 = n - m$, it is shown in [20, Thm. 1] that $d^\mu \sim \frac{n^m}{n!} \cdot d^{\overline{\mu}}$, for $n \gg 0$, where $\overline{\mu} = [\mu_2, \mu_3, \ldots]$, a partition of $n - m$. In particular, by [20, Cor. 2], d^μ always is bounded above by $d^\mu \leq n^m$, while there only is an asymptotical lower bound $n^{m-1} < d^\mu$ for $n \gg 0$. Since for small n there are cases such that the latter inequality fails, the question arises whether there is an explicit bound n_0 such that $n^{m-1} < d^\mu$ for all $n \geq n_0$.

A general strategy to find effective lower bound functions $f(n)$, fulfilling $f(n) \leq d^\mu$ for all $n \geq n_0$ and some explicitly given n_0, is already introduced and used in the proofs of [20, Thm. 5] and its key Lemma [20, La. 4]. The major ingredients to these proofs are the branching rule for irreducible ordinary representations of $\mathcal{S}_n$, see for example [22, Thm. 9.2], the hook length formula [16] for the degree of irreducible ordinary representations, see also [22, Thm. 20.1], and the unitriangularity of the decomposition matrix of $\mathcal{S}_n$. As main theoretical results of the present paper we are going to derive two improvements of [20, La. 4]:

• The first main result, being valid for arbitrary rational primes p, is given in (5.2), whose improvement consists of weaker conditions imposed on candidate lower bound