Journal of Algebra 465 (2016) 399–436

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Homological properties of the homology algebra of the Koszul complex of a local ring: Examples and questions

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ARTICLE INFO

Article history:
Received 3 January 2016
Available online 22 July 2016
Communicated by Jon Carlson

Dedicated to Jörgen Backelin at his 65th birthday

MSC:
primary 13Dxx, 16Dxx, 68W30
secondary 16S37, 55Txx

Keywords:
Local ring
Koszul complex
Gorenstein rings
Yoneda Ext-algebra
Hilbert series
Macaulay2
Programme BERGMAN

ABSTRACT

Let R be a local commutative noetherian ring and HKR the homology ring of the corresponding Koszul complex. We study the homological properties of HKR in particular the Avramov spectral sequence. When the embedding dimension of R is four and when R can be presented with quadratic relations we have found 101 cases where this spectral sequence degenerates and only three cases where it does not degenerate. We also determine completely the Hilbert series of the bigraded Tor of these HKR in Tables A–D at the end of the paper. We also study some higher embedding dimensions. Among the methods used are the programme `BERGMAN` by Jörgen Backelin et al., the `Macaulay2`-package `DGAlgebras` by Frank Moore, combined with results by Govorov, Clas Löfwall, Victor Ufnarovski and others.

0. Introduction and Main Theorem

Let (R, m, k) be a local commutative noetherian ring with maximal ideal m and residue field $k = R/m$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a minimal set of generators of the

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http://dx.doi.org/10.1016/j.jalgebra.2016.06.033
0021-8693/

maximal ideal m. The Koszul complex of R is the exterior algebra $\oplus_{i=0}^{n} \Lambda^i R^n$ of a free R-module of rank n equipped with the differential:

$$d(T_{j_1} \wedge \ldots \wedge T_{j_i}) = \sum_{l=1}^{i} (-1)^{l+1} x_{j_l} T_{j_1} \wedge \ldots \wedge \widehat{T}_{j_l} \wedge \ldots \wedge T_{j_i}$$

It will be denoted by $K(\mathbf{x}, R)$ or KR since it is essentially independent of $\mathbf{x}$. It is a differential graded algebra and its homology algebra *HKR* is a skew-commutative graded algebra over k. In the case that R can be represented as a quotient of a regular local ring $(\tilde{R}, \tilde{m})$ as $R = \tilde{R}/a$, where $a \subset \tilde{m}^2$ (passing to a completion of R we can always assume that this is the case for the problems we are studying), we have that *HKR* is isomorphic to a Tor-algebra:

$$HKR \simeq \mathrm{Tor}_*^{\tilde{R}}(\tilde{R}/a, k)$$

This algebra *HKR* has been studied in various special cases by many authors: it is an exterior algebra if and only if R is a local complete intersection; it is a Poincaré duality algebra if and only if R is a Gorenstein local ring [2] and if R is a Golod ring then the square of the augmentation ideal (*i.e.* the ideal generated by the generators of positive degree) of *HKR* is 0 (the converse is however not true, even for rings with monomial relations [9]). But the general structure of *HKR* (and in particular its homological properties) can be very complicated, even if R is a Koszul ring as we will see below. The aim of the present paper is to combine different methods towards studying *HKR*. We will illustrate our methods on very precise examples, thereby discovering some new unexpected phenomena. We note that *HKR* is graded, and one of our aims is to determine the double series

$$\Phi_R(x, y) = \sum_{p \geq 0, q \geq 0} |\mathrm{Tor}_{p,q}^{HKR}(k, k)| x^p y^q \tag{1}$$

(where $|V|$ denotes the dimension of a k-vector space V) for most (probably essentially all) quadratic rings R of embedding dimension 4: Let $P_R(z) = \sum_{n \geq 0} |\mathrm{Tor}_n^R(k, k)| z^n$. We will see that with the exception of three explicit cases we have $\Phi_R(z, z) = P_R(z)/(1+z)^4$. This last equality can be expressed by saying that the Avramov spectral sequence *degenerates i.e.* degenerates from the second page. Recall that this spectral sequence was introduced in [3] and further studied in [4, formula (6.2.1)]. It is as follows for any local ring R

$$E_n^2 = \bigoplus_{p+q=n} E_{p,q}^2 = \bigoplus_{p+q=n} \mathrm{Tor}_{p,q}^{HKR}(k, k) \Longrightarrow \mathrm{Tor}^R(k, k) \otimes_{KR \otimes k} k$$

In particular there is a coefficientwise inequality $\ll$ of formal power series